A BRIEF INTRODUCTION TO CHARACTERISTIC VARIETIES (WITH EMPHASIS ON LINE ARRANGEMENTS)

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ABSTRACT. Characteristic varieties are very important topological invariants. They generalized Alexandar polynomials. In this short note, I will give a very brief introduction to characteristic varieties and calculations.

1. CHARACTERISTIC VARIETIES: DEFINITIONS

1.1. **Characteristic varieties of modules.** Let M be a finitely generated module over a commutative ring R. Assume that M is also finitely presented. Then we have a finite free presentation

$$R^p \xrightarrow{\varphi} R^q \to M \to 0$$

Let Φ be the matrix representing φ. The i-th Fitting ideal $F_i(M)$ is an ideal of R generated by the $(n-i+1)\times(n-i+1)$ minors of Φ. The reduced subscheme defined by $F_i(M)$ in $\operatorname{Spec} R$ is called the i-th characteristic variety of M, denoted by $\mathrm{Char}_i(M)$. In other words, the i-th characteristic variety of M is the support of the R-module $R/F_i(M)$, denoted by $\mathrm{Supp}(R/F_i(M))$. Let $\mathrm{Ann} \wedge^i M$ be the annihilator of the i-th exterior product of M. A result of Buchsbaum and Eisenbud [BE77] says that $(\mathrm{Ann} \wedge^i M)^q \subset F_i(M) \subset \mathrm{Ann} \wedge^i M$. In particular, $\mathrm{Supp}(R/F_i(M)) = \mathrm{Supp}(\wedge^i M)$. This implies that the definition of characteristic varieties are independent with the presentation. Moreover, we have the following:

Note that characteristic varieties of a R-module M form a descending chain

$$\operatorname{Spec} R \supseteq \mathrm{Char}_1(M) \supseteq \cdots \supseteq \mathrm{Char}_q(M) \supseteq \emptyset.$$

Lemma 1.1. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. Then $\mathrm{Char}_1(M) = \mathrm{Char}_1(M') \cup \mathrm{Char}_1(M'')$ and for $i \geq 2$,*

$$\mathrm{Char}_i(M'') \subset \mathrm{Char}_i(M) \subset \mathrm{Char}_i(M'') \cup \mathrm{Char}_{i-1}(M'') \cap \mathrm{Char}_1(M').$$

Proof. The first equality follows from the exactness of localization. The second follows localization and the following resolution of exterior product

$$0 \to S^k M' \to S^{k-1}M' \otimes \bigwedge^1 M \to \cdots \to S^1 M' \otimes \bigwedge^{k-1} M \to \bigwedge^k M \to \bigwedge^k M'' \to 0.$$

□

Example 1.1 (Characteristic variety associated to a group G)**.** *Let G be a finitely generated and finitely presented group. Denote by G' and G'' the first and second commutator groups. Since G'/G'' is abelian, there is a natural action of G/G' on G'/G'' given by $\phi(b)a\phi(b)^{-1}$, where $b \in G/G''$, $a \in G'/G''$ and $\phi : G/G'' \to G/G'$ is the surjective morphism. We can extend the action linearly and define a $\mathbb{Z}[G/G']$-module structure on G'/G''. Crowell [Cro61] shows that there is a unique exact sequence of $\mathbb{Z}[G/G']$-modules*

$$0 \to G'/G'' \to A \to \mathbb{Z}[G/G'] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

associated to the exact sequence of groups

$$1 \to G'/G'' \to G/G'' \to G/G' \to 1,$$

where ε is the trivial morphism. Characteristic varieties of the $\mathbb{Z}[G/G']$-module of G'/G'' are important invariants of G, especially when G is a fundamental group.

1.2. **Characteristic varieties of topological spaces.** Let X be the topological space such that the fundamental group $G = \pi_1(X)$ is a finitely generated and finitely presented group such that $H_1(X, \mathbb{Z}) = \mathbb{Z}^r$, for instance, complements of plane algebraic curves. Let $\tilde{X} \to X$ be the universal Abelian covering of X corresponding to the abelianization morphism $ab : \pi_1(X) \to H_1(X, \mathbb{Z})$. The deck transformation group $\mathrm{Aut}(\tilde{X}/X) = H_1(X, \mathbb{Z})$ defines an action on $G'/G'' = H_1(\tilde{X}, \mathbb{Z})$ which is the same as in Example 1.1. By extending linearly to the group ring $R = \mathbb{C}[K] = \mathbb{C}[t_1^{\pm}, \cdots, t_r^{\pm}]$, called Laurent polynomial ring, this action defines a R-module structure on $H_1(\tilde{X}, \mathbb{C})$. We call this R-module $H_1(\tilde{X}, \mathbb{C})$ the Alexander invariant of X. It is clear that this invariant depends only on $\pi_1(X)$. Denote by $F_i^{\mathbb{C}}(X)$ be the i-th Fitting ideal of $H_1(\tilde{X}, \mathbb{C})$.

Definition 1.2 (Characteristic Varieties)**.** The i-th characteristic variety, denoted by $\mathrm{Char}_i(X)$ of X is the reduced subvariety of $\mathrm{Spec} R = (\mathbb{C}^*)^r$ defined by the i-th Fitting ideal $F_i^{\mathbb{C}}(X)$.

Example 1.2. *Let $\tilde{X} \to X = \vee_r S^1$ be the universal abelian cover corresponding to $\pi_1(X) = \mathbb{F}_r \to \mathbb{Z}^r \to 0$. To motivate the general situation, we first consider the case $r = 1$. In this case, $\tilde{X}$ is nothing but the universal cover, since $\pi_1(S^1) = \mathbb{Z}$. We have a natural way to write down a chain complex of $\tilde{X}$, i.e. $C_0(\tilde{X}) = \mathbb{Z}[\mathbb{Z}]$ and $C_1(\tilde{X}) = \mathbb{Z}[\mathbb{Z}]$. Apply the construction to each circle such that they have the same origin buy independent to each other, we obtain a r-dimensional grid $\mathbb{Z}^r$ which is the universal abelian cover of $\tilde{X}$. Then the 0-complex $C_0(\tilde{x}) = \mathbb{Z}[\mathbb{Z}^r]$ and the generating 1-simplices are liftings of generators of $\pi_1(X)$. Hence $C_1(\tilde{X}) = (\mathbb{Z}[\mathbb{Z}^r])^r$, and the boundary morphism $\partial_1 : C_1(\tilde{X}) \to C_0(X)$ is $\partial(\tilde{x}_i) = x_i - 1$, where $\tilde{x}_i$ is a basis of*

$C_1(X)$ *and* x_i *is the end point of* $\tilde{x}_i$ *in* $C_0(\tilde{X})$. *Therefore the Alexander invariant* $H_1(\tilde{X}, \mathbb{C})$ *fits in the following short exact sequence* R*-modules*

$$0 \to H_1(\tilde{X}, \mathbb{C}) \to R^r \xrightarrow{\partial_1} I \to 0,$$

where $I =< x_1 - 1, x_2 - 1, \cdots, x_r - 1 >\subset R$. *On the other hand, we have the Koszul resolution of* $\mathbb{C}$ *as* R*-modules*

$$(*) \qquad 0 \to \bigwedge^{r} R \xrightarrow{d_r} \bigwedge^{\binom{r}{r-1}} R \xrightarrow{d_{r-1}} \cdots \xrightarrow{d_4} \bigwedge^{\binom{r}{3}} R \xrightarrow{d_3} \bigwedge^{\binom{r}{2}} R \xrightarrow{d_2} \bigwedge^{r} R \xrightarrow{\partial_1} R \xrightarrow{\varepsilon} \mathbb{C} \to 0.$$

Comparing the exact sequences, we see that $H_1(\tilde{X}, \mathbb{C})$ *has a presentation*

$$\bigwedge^{\binom{r}{3}} R \xrightarrow{\partial_3} \bigwedge^{\binom{r}{2}} R \to H_1(\tilde{X}, \mathbb{C}) \to 0.$$

This implies that $\mathrm{Char}_i(X) = (\mathbb{C}^*)^r$ *for* $0 \le i \le r-1$, $\mathrm{Char}_i(X) = \mathbf{1}$ *for* $n \le i \le \binom{r}{2}$ *and* $\mathrm{Char}_i(X) = \emptyset$ *for* $i > \binom{r}{2}$.

In general we are not so lucky to get a presentation of $H_1(\tilde{X}, \mathbb{C})$ directly. Recall that, by [Cro61], $H_1(\tilde{X}, \mathbb{C})$ fits in a exact sequence of R-modules

$$0 \to H_1(\tilde{X}, \mathbb{C}) \to A \to \mathbb{C}[\mathbb{Z}^r] \xrightarrow{\varepsilon} \mathbb{C} \to 0.$$

On the other hand, since regular covering $\tilde{X} \to Z$ are principal $\mathrm{Aut}(\tilde{X}/X)$-bundles, we have the following exact sequence

$$0 \to H_1(\tilde{X}, \mathbb{C}) \to H_1(\tilde{X}, \phi^{-1}(b), \mathbb{C}) \to H_0(\phi^{-1}(b), \mathbb{C}) \to H_0(\tilde{X}, \mathbb{C}) \to H_0(\tilde{X}, \phi^{-1}(b), \mathbb{C}) \to 0.$$

It is not hard to see that $H_0(\phi^{-1}(b), \mathbb{C}) \to H_0(\tilde{X}, \mathbb{C})$ is the same as $\mathbb{C}[\mathbb{Z}^r] \xrightarrow{\varepsilon} \mathbb{C}$. Therefore, we obtain a shorter one.

$$0 \to H_1(\tilde{X}, \mathbb{C}) \to H_1(\tilde{X}, \phi^{-1}(b), \mathbb{C}) \to \mathbb{C}[\mathbb{Z}^r] \xrightarrow{\varepsilon} \mathbb{C} \to 0.$$

Comparing the two exact sequence, we see that $A = H_1(\tilde{X}, \phi^{-1}(b), \mathbb{C})$. Denote by I there kernel of ε, we get the following short exact sequence of R-modules

$$0 \to H_1(\tilde{X}, \mathbb{C}) \to H_1(\tilde{X}, \phi^{-1}(b), \mathbb{C}) \to I \to 0.$$

It is easy to see that $I =< x_1 - 1, x_2 - 1, \ldots, x_r - 1 >\subset R$, where x_i are generators of $H_1(X, \mathbb{Z})$. This sequence is called Crowell exact sequence.

The R-module $H_1(\tilde{X}, \phi^{-1}(b), \mathbb{C})$ in the middle of Crowell exact sequence is called Alexander module of X. Using Fox calculus, one can find a presentation of the Alexander module.

2. Fox derivatives

Fox derivatives are very useful to give the boundary maps in a chain complexes

$$\to C_2(\tilde{X}) \xrightarrow{\partial_2} C_1(\tilde{X}) \xrightarrow{\partial_1} C_0(\tilde{X}) \to 0.$$

In Example 1.2, we have implicitly used the idea.

Example 2.1. *Let $X = \vee_2 S^1$ and $\tilde{X}$ be the universal covering of X. Then $\tilde{X}$ has a homotopy model, the Caylay tree of $\mathbb{F}_2$. Let $\tilde{x}$ and $\tilde{y}$ be the generators (the two directions) of the tree. Then the lifting of the product xy can be written as $\widetilde{xy} = \tilde{x} + x\tilde{y}$, where x is the end point of $\tilde{x}$, and $x\tilde{y}$ means to shift the initial point (i.e. the origin of the tree) of $\tilde{y}$ to x. This gives a basic rule for lifting paths into covering space. Using this fact, we can view $C_1(\tilde{X})$ as a $\mathbb{Z}[\mathbb{F}_2]$-module generated by $\tilde{x}$ and $\tilde{y}$. Moreover, we obtain a chain complex*

$$0 \to C_1(\tilde{x}) \xrightarrow{\partial_1} C_0(\tilde{X}) = \mathbb{Z}[\mathbb{F}_2] \to 0,$$

where $\partial_1(\tilde{x}) = x - 1$ and $\partial_1(\tilde{y}) = y - 1$.

More generally, we can also this rule to determine the second boundary map. Let $G = \pi_1(X) =< x_1, \cdots x_m | r_1, \cdots r_n >$ be the presentation of a fundamental group of a topological space X. We can construct a homotopy model from the wedge of m circles by imposing 2-cells with boundary is r_i. Let $\tilde{s_i}$ be the lifting of s_i into the universal space $\tilde{X}$, then $\partial_2(\tilde{s_i}) = \tilde{r_i}$. Therefore, ∂_2 can be determined by the lifting rule in Example 2.1. The lifting rule called Fox differential.

Definition 2.1. Let $F =< x_1, \dots, x_m >$ be a free group. A Fox derivative of the group ring $\mathbb{C}[F]$ with respect to a generator x_i of F is a $\mathbb{C}$-linear map $\frac{\partial}{\partial x_i} : \mathbb{C}[F] \to \mathbb{C}[F]$ such that:

(1) $\frac{\partial}{\partial x_i}(ab) = \frac{\partial}{\partial x_i}a + a\frac{\partial}{\partial x_i}b$ for any $a, b \in \mathbb{C}[F]$;

(2) $\frac{\partial}{\partial x_i}(c) = 0$ for any $c \in \mathbb{C}$;

(3) $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol.

Example 2.2.

(1)
$$\frac{\partial}{\partial x_i}(x_i^{-1}) = -x_i^{-1}.$$

(2) $$\frac{\partial}{\partial x_i}(x_i x_j x_i^{-1}) = \frac{\partial}{\partial x_i}(x_i) + x_i \frac{\partial}{\partial x_i}(x_j x_i^{-1}) = 1 + x_i x_j \frac{\partial}{\partial x_i}(x_i^{-1}) = 1 - x_i x_j x_i^{-1}.$$

$$\frac{\partial}{\partial x_j}(x_i x_j x_i^{-1}) = x_i \frac{\partial}{\partial x_j}(x_j x_i^{-1}) = x_i(1 + x_j \frac{\partial}{\partial x_j}(x_i^{-1})) = x_i. \tag{3}$$

Come back to the group $G = \pi_1(X) =< x_1, \cdots x_m | r_1, \cdots r_n$. The second boundary map ∂_2 is determine by the Jacobian matrix

$$\left(\frac{\partial r_i}{\partial x_j}\right)^{\psi} = \left(\psi(\frac{\partial r_i}{\partial x_j})\right),$$

where $\psi : \mathbb{F}_m \to G$ is the canonical quotient map. More generally, for any regular covering $X' \to X$ corresponding to a surjective morphism $\varphi : \pi_1(X) \to K$, we define the Jacobian matrix as $\left(\frac{\partial r_i}{\partial x_j}\right)^{\varphi\circ\psi}$.

In particular, for the universal abelian covering $\tilde{X} \to X$ with $H_1(X) = \mathbb{Z}^r$, we have the following augmentation chain complex of $\tilde{X}$.

$$\cdots \to \mathbb{Z}[\mathbb{Z}^r]^n \overset{\partial_2}{\to} \mathbb{Z}[\mathbb{Z}^r]^m \overset{\partial_1}{\to} \mathbb{Z}[\mathbb{Z}^r] \overset{\mu}{\to} \mathbb{Z} \to 0,$$

where $\partial_2 = \left(\dfrac{\partial r_i}{\partial x_j}\right)^{ab\circ\phi}$.

Comparing this chain complex with the Crowell exact sequence, we get a presentation of the Alexander module, instead of the Alexander invariant,

$$\mathbb{Z}[\mathbb{Z}^r]^n \overset{\partial_2}{\to} \mathbb{Z}[\mathbb{Z}^r]^m \to A \to 0.$$

By comparing with the Koszul resolution of $\mathbb{C}$ in the category of R-modules. One may get a presentation of the Alexander invariant. For more discussion, see [CS99] .

3. Homology of Abelian Covers

The first betti numbers of abelian covers of complements of algebraic curves has very closed relation with characteristic varieties. In this section, I will give a very brief introduction to Libgober's algorithm on computation of characteristic varieties of complements of plane algebraic curves.

Let $X = \mathbb{C}^2 \setminus C$ be the complement of a plane algebraic curve C. Let $\varphi_{\vec{m}} : \pi_1(X) \to \mathbb{Z}/(m_1) \oplus \mathbb{Z}/(m_2) \oplus \cdots \oplus \mathbb{Z}/(m_r)$. Denote by $X_{\vec{m}} \to X$ the abelian cover defined by $\varphi_{\vec{m}}$. Then $X_{\vec{m}}$ defines a birational class of projective surfaces $\overline{X_{\vec{m}}}$. Birational invariants of $\overline{X_{\vec{m}}}$ depend only on C and the homomorphism $\varphi_{\vec{m}}$.

Remark 3.1. The first betti number is a birational invariant. By Hodge theory, $b_1 = h^{1,0} + h^{0,1} = 2h^{0,1} = 2\dim H^0(\Omega)$. Birational varieties are isomorphic outside of a real 2-dimensional closed subvarietie. By Hartogs theorem, p-forms extends. Therefore $h^{0,p}$ is birational invariant.

Since the first betti number is a birational invariant, we can pick up a birational model of $\overline{X_{\vec{m}}}$ to study. A natural one is the complete intersection $V_{\vec{m}}$ in $\mathbb{P}^{r+2}$ (coordinates are denoted by $z_1, \dots, z_r, u, x, y$) of r equations

$$z_1^{m_1} = u^{m_1-d_1} f_1(u,x,y), \dots, z_1^{m_r} = u^{m_r-d_r} f_r(u,x,y)$$

where $f_1, f_2, \dots, f_r$ are the equations of irreducible components of C and $d_i = \deg f_i$.

A theorem of Sakuma (see [Lib01], [Sak95]) tells us how to compute the first betti number of a smooth resolution of $V_{\vec{m}}$ in terms of characteristic varieties of irreducible components. Let $W \to V_{\vec{m}}$ be a resolution. For a torsion point $\omega = (\omega_1, \dots, \omega_r) \in (\mathbb{C}^*)^r$, let $C_\omega = \cup_{i,\omega_i \neq 1} C_i$ and χ_ω be a character of $\mathrm{Aut}(W/\mathbb{P}^2)$ such that $\chi_\omega(\gamma_i) = \omega_i$, where γ_i are generators of $\mathrm{Aut}(W/\mathbb{P}^2)$.

Theorem 3.2. *The first Betti number of W equals*

$$\sum_\omega \max\{i \mid \omega \in \mathrm{Char}_i(C)\}. \tag{4}$$

More precisely,

$$\dim H_{1,\chi_\omega}(W) = \max\{i \mid \omega \in \mathrm{Char}_i(C)\},$$

where

$$H_{1,\chi_\omega}(W) = \{x \in H_1(W) \mid g(x) = \chi_\omega(g)\cdot x,\ \ \forall g \in \mathrm{Aut}(W/\mathbb{P}^2)\}.$$

We shall call $\dim H_{1,\chi_\omega}(W)$ the multiplicity of the character χ_ω.

On the other hand, Libgober provides another way to calculate the multiplicity of a character χ_ω which allows us to determine if a character is in a characteristic variety.

Definition 3.3. Let F be a complete intersection of $n-2$ hypersurfaces in $\mathbb{P}^n$ and $f : \tilde{F} \to F$ be a resolution. We define the adjoint idea $Adj(F)$ of F as $\pi^{-1}(f_*(\omega_{\tilde{F}/F}))$, where $\pi : \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_F$ is the quotient morphism.

Let $f = f_1 \cdots f_r$ be an germ of a reduced algebraic curve with irreducible components $f_i = 0$ at the origin. Given a vector $(m_1, \dots, m_r)$, we denote by $V_{\vec{m}}$ be the complete intersection of $z^{m_1} = f_1, \dots, z^{m_r} = f_r$ in $\mathbb{C}^{r+2}$. Let A be an ideal in the local ring $\mathcal{O}_{\mathbb{C}^{r+2},\mathbf{0}}$.

Definition 3.4 (Ideal of quasiadjunction)**.** We call A the ideal of quasiadjunction of f with parameters $(\vec{m} \mid \vec{j})$, where $\vec{j} = (j_1, \dots, j_r)$, if $A = \{\phi \mid z^{j_1} \cdots z^{j_r}\phi \in Adj(V_{\vec{m}})\}$. An ideal of quasiadjunction is an ideal of qusiadjunction for some parameters.

The following proposition give a combinatorial characterization of ideals of quasiadjunction.

Proposition 3.5. *An ideal of A of quasiadjunction is an ideal of quasiadjunction of f with parameter $(\vec{m} \mid \vec{j})$ if and only if $(\frac{j_1+1}{m+1}, \ldots, \frac{j_r+1}{m_r}) \in \bar{\Delta}(A)$, where $\Delta(A)$ is a open polytope in the unit cube determined by the inequalities:*

$$\sum_{i=1}^{r} a_{k,i} x_i > \sum_{i=z}^{r} a_{k,i} - f_k(A) - c_k - 1$$

for all k, where $a_{k,i} = \mathrm{mult}_{E_k} f^*(f_i)$, $c_k = \mathrm{mult}_{E_k} f^*(\mathrm{d}x \wedge \mathrm{d}y)$, $f_k(A) = \min \{\mathrm{mult}_{E_k} f^*(\phi) \mid \phi \in A\}$, and E_k are the exceptional divisors on the resolution $f : W \to V_{\vec{m}}$.

Ideals of quasiadjunction are determined by the singularity. To demonstrate the algorithm of calculation of characteristic varieties, we will focus arrangements of n lines.

Proposition 3.6. *Let L_1, L_2, …, L_m be line which intersect at the origin. Then there are finitely many ideals of quasiadjunction which are $\mathfrak{m}^r$, where $r = 1, 2, \ldots, m-2$. And the corresponding polytopes are $x_1 + \cdots + x_m > m - 1 - r$.*

The hyperplanes $x_1 + \cdots + x_m = m - 1 - r$ are called faces of ideals of quasiadjunction.

Now we can talk about the algorithm for line arrangements in a very rough way.

(0) Determine faces of local polytopes of all singular points

(1) Take all subsets δ of set of points, such that summation of faces

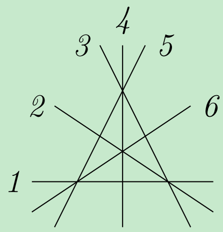
$$L_P = x_{i_1} + \cdots + x_{i_m} = m - 1 - r = s_P$$

of local polytopes of points in δ equivalent to $x_1 + x_2 + \cdots + x_n = l$. We call $l(\delta) = l$ the level of δ. Such a subset δ and corresponding polytopes are said to be contributing.

($1\frac{1}{2}$) Calculate $\dim H^1(\mathscr{A}_\delta(n-3-l(\delta)))$, where $\mathscr{A}_\delta$ is the sheaf of ideals of quasiadjunction corresponding to δ.

(2) Test contributing face. If $k = \dim H^1(\mathscr{A}_\delta(n-3-l(\delta))) > 0$. Then the equations $\exp(2\pi\sqrt{-1}L_P) = \exp(2\pi\sqrt{-1}\beta_P)$ (using Sakuma's theorem) defines an essential component of characteristic varieties Char_k. In other words, the essential components of the characteristic varieties are Zariskis closures of the images of the contributing faces under the exponential map.

Example 3.1. *Let $\mathscr{A}$ be the Ceva arrangement (see Figure)*



The faces corresponding to the four triple points are

$$x_1 + x_2 + x_3 = 1, x_3 + x_4 + x_5 = 1, x_2 + x_4 + x_6 = 1, x_1 + x_5 + x_6 = 1.$$

The corresponding ideas of quasiadjunction are the maximal ideals. The set of the four triple points is the only set δ that the summation of faces equivalent to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = k$, where k has to be 2. Let $t_i = \exp(2\pi\sqrt{-1}x_i)$. Then a possible essential component is the intersection of $t_1t_2t_3 = 1$, $t_3t_4t_5 = 1$, $t_2t_4t_6 = 1$ and $t_1t_5t_6 = 1$. We have to calculate $\dim H^1(\mathscr{A}_\delta(6-3-2))$. Notice that the four triple points are on the complete intersection of two quadrics. Using the following exact sequence $0 \to \mathscr{O}(-4) \to \mathscr{O}(-2) \oplus \mathscr{O}(-2) \to \mathscr{A}_\delta \to 0$, we see that $h^1(\mathscr{A}_\delta(6-3-2)) = h^2(\mathscr{O}(-3)) = h^0(\mathscr{O}) = 1$. Therefore, this is an essential component of Char_1.

Let us see another example.

A basic ideal of a singularity $f_1 f_2 \cdots f_r = 0$ is the ideal $A(f_1, f_2, \ldots, f_r)$ generated by

$$\frac{(f_i)_x}{f_i} f_1 f_2 \cdots f_r, \frac{(f_i)_y}{f_i} f_1 f_2 \cdots f_r, \frac{Jac(\frac{(f_i, f_j)}{(x,y)})}{f_i f_j} f_1 f_2 \cdots f_r.$$

Then any ideal of quasi adjunction contains $A(f_1, f_2, \ldots, f_r)$. The local ring $\mathscr{O}$ is considered as an "improper ideal" of quasiadjunction.

Example 3.2. *Let $\mathscr{A}$ be the conic-line arrangement defined by $x(x-2)y(y-2)(x^2+y^2-2x-2y) = 0$. Then $\mathscr{A}$ has 4 triple points (see Figure 1). At each triple points, the basic ideal is the maximal ideal. Since each triple point is a normal crossing, a single blowing-up will resolve it. Therefore, the face of a local quasiadjunction is of the form $x_{i_1} + x_{i_2} + x_{i_3} = 1$. Let x_1 be the variable associated to the conic, and x_{i+1} be the variables associated to L_i for $i = 1, 2, 3, 4$. There is only one contributing face $\delta : 2x_1 + x_2 + \cdots + x_5 = 2$. To see if the image of the contributing face under the exponential map contains elements of the characteristic varieties, we need to compute*

$$\dim H^1(\mathbb{P}^2, \mathscr{A}_\delta(d-3-l(\delta))).$$

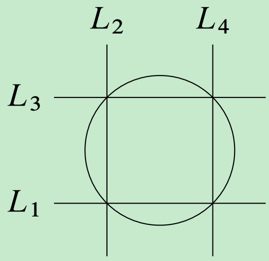


FIGURE 1

In this case, $\mathscr{A}_\delta = I_{\text{triple points}} = (x(x-2)y(y-2), x^2+y^2-2x-2y)$*,* $d=6$*, and* $l(\delta)=2$*. Since* $\mathscr{A}_\delta$ *is a complete intersection. Then, we have the following exact sequence*

$$0 \to \mathscr{O}(-6) \to \mathscr{O}(-2) \oplus \mathscr{O}(-4) \to \mathscr{A}_\delta \to 0.$$

We see that $h^1(\mathscr{A}_\delta(6-3-2)) = h^0(\mathscr{O}(3)) - h^0(\mathscr{O}(1)) = 6$*. Therefore, characteristic varieties of this conic-line arrangement has an essential component:*

$$\{(t_1, t_2, \ldots, t_6) \in (\mathbb{C}^*)^6 \mid t_1t_2t_3 = 1, t_1t_3t_4 = 1, t_1t_4t_5 = 1, t_1t_2t_5 = 1.\}$$

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