NOTE ON GEOMETRY OF COHEN-MACAULAY RINGS

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This is a personal note to understand Cohen-Macaulayness. All rings are assumed to be commutative Noetherian rings with identity. Main references are [Hoc78] [Hoc80] [Kun85], [Eis95]. Discussions on the MO question Geometric meaning of Cohen-Macaulay schemes are also quite inspiring. We always denote by R a ring and M an R-module.

Warning: there might be typos and mistakes. Please let me know if you see one. Thanks!

1. Zero Divisors

We have to recall some useful results on zero divisors of a ring or module.

Definition 1.1. A prime ideal P is said to be an ***associated prime ideal*** of M if there exists an $m \in M$ such that $P = \mathrm{Ann}(m)$. The set of associated prime ideals of M is denoted by $\mathrm{Ass}(M)$.

Clearly if $P = \mathrm{Ann}(m)$, then $mR \cong R/P$ and $\mathrm{Ann}(M) \subset P$ for any $P \in \mathrm{Ass}(M)$.

Remark 1.2.

- The zero ideal (0) is not always prime. It is prime if and only if the ring is a domain. As a corollary, it is easy to see that an ideal $P \subset R$ is prime if and only if R/P is a domain.
- The whole ring R is not considered as a maximal ideal of itself.

Proposition 1.3. *Assume that* $M \neq 0$*, then* $\mathrm{Ass}(M) \neq \emptyset$.

Proof. Let S be the set of ideal of annihilators of elements of M. Since we assume that R is Noetherian. Then there is an ideal $I = \mathrm{Ann}(m) \in S$ which is maximal in S. Let $ab \in I$. We want to show that either $a \in I$ or $b \in I$. Assume that $b \notin I$. Then $a \in \mathrm{Ann}(bm)$. Since I is maximal, then $(a) + I = I$. □

Lemma 1.4. *Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence of* R*-modules. Then* $\mathrm{Ass}(M') \subset \mathrm{Ass}(M) \subset (\mathrm{Ass}(M') \cup \mathrm{Ass}(M''))$.

Proof. The first containment is clear. For the second, assume that $P \in \mathrm{Ass}(M) \setminus \mathrm{Ass}(M')$. Then there is a $m \in M$ but $m \notin M'$ such that $mR \cong R/P$. Note that $mR \cap M' = (0)$, therefore, the mR is isomorphic to its image $m''R$ in M'', where m'' is the image of m in M''. Thus $P = \mathrm{Ann}(m'') \in \mathrm{Ass}(M'')$. □

Lemma 1.5. *For any prime ideal* P*, we have* $\mathrm{Ass}(R/P) = \{P\}$.

Proof. Let Q be a prime ideal in $\mathrm{Ass}(R/P)$. Then $P = \mathrm{Ann}(R/P) \subset Q \subset P$. Here we used the fact the P is prime. Therefore $Q = P$. □

Proposition 1.6. *The set of zero divisor of M is* $\bigcup_{P \in \mathrm{Ass}(M)} P$.

Proof. It suffice to show that a nonzero element $r \in R$ is in some associated prime ideal P if it is a zero divisor of M. Let $m \in M$ be an nonzero element such that $rm = 0$. Note that $\mathrm{Ass}(mR)$ is nonempty since $mR \neq (0)$. Therefore, there is a prime ideal $P \in \mathrm{Ass}(mR)$ such that $P = \mathrm{Ann}(rm)$ for some $r \in R$. Since $r(r'm) = r'(rm) = 0$, therefore $r \in \mathrm{Ann}(r'm) = P$. □

Proposition 1.7. *Assume that M is finitely generated. Then there is a chain of submodules*

$$M = M_0 \supset M_1 \supset M_2 \cdots \supset M_n = (0)$$

such that $M_i/M_{i+1} \cong R/P_i$ for some prime ideal P_i for $i = 0, 1, 2, \ldots, n-1$.

Proof. Let S be the set of non-zero submodules of M for which the proposition holds. Assume that $M \neq 0$. Then S is nonempty, since there is a $P \in \mathrm{Ass}(M)$ such that $R/P \cong mR$ for some $m \in M$. Since M is finitely generate, then S has a maximal element N. If $M/N \neq 0$, then M/N contains a nonzero submodule which is isomorphic to R/P for some prime ideal P. Let N' be the inverse image of N' in M. Then $N'/N \cong R/P$, contradicting the maximal property of N. Therefore $M = N$ and the proposition is proved. □

Corollary 1.8. *Assume that M is finitely generated. Then $\mathrm{Ass}(M)$ is a finite set.*

Proof. It follows from Lemma 1.4, Lemma 1.5 and Proposition 1.7. □

Lemma 1.9 (Prime Avoidance). *Let $I_1, I_2, \ldots, I_n$ and J be ideal of a ring R. Assume that $J \subseteq \cup_{j=1}^{n} I_j$. If R contains an infinite field or if at most two of the I_j are not prime, then J is contained in one of the I_j.*

Proof. If R contains an infinite field, then The result is trivial due to the following two observations:

- each ideal is a vector space over the infinite field.
- No vector space over an infinite field is a finite union of proper subspaces.

In the other case, we do induction on n. The case $n = 1$ is clear. We may assume that $J \not\subseteq \cup_{j \neq k} I_j$ for all $k = 1, 2, \ldots, n$. Otherwise, we can apply induction. For $n = 2$, let $x_1 \in J \setminus I_2$ and $x_2 \in J \setminus I_1$, then $x_1 + x_2$ is in J, but it is neither in I_1 nor I_2. For $n > 2$, at least one of the I_k is prime, we may assume that I_1 is prime. Set $y = x_1 + x_2 x_3 \cdots x_n$, where $x_k \in J \setminus \cup_{j \neq k} I_j$. Again $y \in S$, but $y \notin I_k$ for $k = 1, 2, \ldots, n$. In fact, if $y \in I_1$, then $x_2 x_3 \cdots x_n \in I_1$. Since I_1 is prime, then $x_i \in I_1$ for some $i > 1$. But $x_i \in S \setminus \cup_{j \neq k} I_j \subset S \setminus I_1$. If $y \in I_j$ for some $j > 1$, then $x_1 \in I_j$ □

Theorem 1.10. *Assume that M is finitely generated. If I is an ideal consists of zero divisors of M, then there exists an $m \in M$ such that $mI = (0)$.*

Proof. It follows from Proposition 1.6 and prime avoidance theorem. □

2. Cohen-Macaulay Rings: Definitions and Basic Properties

There are many ways to define "dimensions". Two most important and natural "dimensions" are Krull dimension and depth. Roughly speaking, Krull dimension measures how big a topological space can be embedded into the variety; depth measures how many times one cut out the variety properly by hypersurfaces.

Definition 2.1. For a ring R (prime ideal $P \subset R$), the ***Krull dimension*** (or simply dimension) $\dim R$ of a ring R (respectively ***height*** height$P$ of a prime ideal P) is defined as the supremum of lengths r of chains of prime ideals

$$P_r \supsetneq R_{r-1} \supsetneq \cdots \supsetneq P_0$$

(respectively $P = P_r$). The height of an proper ideal I of R is defined as

$$\text{height } I = \min\{\text{height } P \mid I \subset P, P \text{ prime}\}.$$

Note that R is never regard as prime. The zero ideal is prime if and only if R is integral.

Remark 2.2.

- Nagata (Example 1, Appendix A1 [Nag62]) provided an example of Noetherian ring which has infinite Krull dimension.
- Nagata (Example 2, Appendix A1 [Nag62]) also provided an example shows that not every chain can be extended to a maximal chain and maximal chains of prime ideals may not have the same length.
- Rings in which every chain of prime ideals can be extended to a maximal chain are known as ***catenary rings***.

Definition 2.3. Let R be a ring and M be a R-module. A sequence of nonzero elements $x_1, x_2, \ldots, x_r \in R$ is called a ***regular sequence*** on M (or simply M-sequence) if

(1) $(x_1, x_2, \ldots, x_r)M \neq M$;

(2) For $i = 1, 2, \ldots, r$, x_i is not a zero divisor of $M/(x_1, x_2, \ldots, x_{i-1})M$.

Remark 2.4. Since R is Noetherian, the maximal length of M-sequences exists. This follows easily from the fact that $(x_1), (x_1, x_2), \ldots, (x_1, x_2, \ldots, x_r)$ forms a properly ascending chain of ideals.

Remark 2.5. Regular sequences are sensitive to the order. Consider $R = k[x, y, z]$. Then $x, y(1-x), z(1-x)$ is an R-sequence, but $y(1-x), z(1-x), x$ is not.

Example 2.6. Consider $R = k[x, y, z]/(xy)$. Then (z, x-y) is a regular sequence.

Example 2.7. Consider $R = k[x, y, z]/(xy - z)$. Then (z, x-y) is a regular sequence.

In fact the rings in those examples are Cohen-Macauly rings.

Lemma 2.8. *Let I, J be ideals in R, M be an R-module, and $N = M/IM$. Then $N/JN \cong M/(I+J)M$.*

Proof. It suffices to show that $JN = (I+J)M/IM$. However, this is clear. □

Theorem 2.9. *Let M be an R-module and $x_1, \ldots, x_r$ are elements in R. For any $i \in \{1, 2, \ldots, r\}$, the following are equivalent.*

(1) $(x_1, \ldots, x_r)$ is an M-sequence,

(2) $(x_1, \ldots, x_i)$ is an M-sequence and $(x_{i+1}, \ldots, x_r)$ is an $M/(x_1, x_2, \ldots, x_i)M$-sequence.

Proof. Apply Lemma 2.8 to $I = (x_1, \ldots, x_i)$ and J successively replaced by (x_{i+1}), (x_{i+1}, x_{i+2}). □

Lemma 2.10. *If (x, y) is an M-sequence and y is not a zero divisor of M, then (y, x) is also an M-sequence.*

Proof. It suffice to show that x is not a zero divisor of M/yM. Let $m \in M$ be an element such that $xm \in yM$. Then there exists an element $m' \in M$ such that $xm = ym'$. Since (x, y) is a M-sequence. Then $m' \in xM$. Thus there is a $m'' \in M$ such that $m' = xm''$. Since x is not a zero divisor, we can cancel out x and get $m = ym''$. Therefore $m \in yM$. We conclude that x is not a zero divisor of M/yM. □

Corollary 2.11. *If $(x_1, x_2, \ldots, x_r, y)$ is an M-sequence and y is not a zero divisor of any of the modules $M/(x_1, x_2, \ldots, x_i)$ for $i = 0, 1, \ldots, r$. Then $(y, x_1, x_2, \ldots, x_r)$ is also an M-sequence.*

Proof. Apply repeatedly Lemma 2.10 and Theorem 2.9. □

Proposition 2.12. *Let R be a ring, I be an ideal of R and M be an R-module. Assume that $IM \neq M$. Then any two maximal M-sequences in I have the same length.*

The idea of the following proof is due to Nortchcott-Rees [NR57].

Proof. It suffices to show that if $(x_1, x_2, \ldots, x_r)$ is a maximal M-sequence, then an M-sequence $(y_1, y_2, \ldots, y_r)$ is also maximal. The proof can be done by induction on r. When $r = 1$, I contains only zero divisors of M/xM. By Theorem 1.10, there is an $m \in M \setminus xM$ such that $mI = 0$. In particular $my = m'x$ for some $m' \in M$. Since y is not a zero divisor of M, then m' can not be in yM. From $m'xI = myI \subset xyM$, it follows that $m'I \subset yM$. Therefore, all element in I are zero divisor of M/yM.

For $n > 1$, put $A_1 = M/(x_1, \ldots, x_i)M$ and $B_i = M/(y_1, \ldots, y_i)M$ for $i = 0, 1, \ldots, r-1$. We can choose a $z \in I$ that is not a zero divisor of A_i and B_i for all $i = 0, 1, \ldots, r-1$. Because, zero divisors the modules are contained in a finite union of prime ideals and I contains non zero divisors, hence can not be in any of the prime ideals. By apply the argument in the case $n = 1$ to $M/(x_1, x_2, \ldots, x_{r-1})$, we see that $(x_1, \ldots, x_{r-1}, z)$ is maximal because of the maximal property of $(x_1, \ldots, x_{r-1}, x_r)$.

By Corollary 2.11, $(z, x_1, x_2, \ldots, x_{r-1})$ and $(z, y_1, y_2, \ldots, y_{r-1})$ are both M-sequence. Moreover $(z, x_1, x_2, \ldots, x_{r-1})$ is maximal. Then $(x_1, x_2, \ldots, x_{r-1})$ and $(y_1, y_2, \ldots, y_{r-1})$ and both M/zM-sequence in I and $(x_1, x_2, \ldots, x_{r-1})$ is maximal. Hence $(y_1, y_2, \ldots, y_{r-1})$ is maximal by induction. Therefore, $(z, y_1, y_2, \ldots, y_{r-1})$ is a maximal M-sequence, so is $(y_1, y_2, \ldots, y_{r-1}, z)$. By the case $n = 1$ again, we show that $(y_1, y_2, \ldots, y_{r-1}, y_r)$ is maximal. □

Definition 2.13. The ***depth*** of M in an ideal I is

$$\operatorname{depth}_I M = \max\{r \mid (x_1, x_2, \ldots, x_r) \text{ is a } M\text{-sequence in } I\}.$$

Corollary 2.14. *If* $(x_1, x_2, \ldots, x_r)$ *is an* M*-sequence, then* $(x_1^{t_1}, x_2^{t_1}, \ldots, x_r^{t_r})$ *is an* M*-sequence for any positive integers* $t_1, t_2, \ldots, t_r$*. In particular,* $\operatorname{depth}_I M = \operatorname{depth}_{I^m} M$.

Remark 2.15. Proposition 2.12 also has a proof using homological method. More precisely, apply the following theorem with $N = R/I$, we get Proposition 2.12.

Theorem 2.16. *Let* M *and* N *be two* R*-modules. Assume that* $x_1, x_2, \ldots, x_r$ *is an* M*-sequence and* $(x_1, x_2, \ldots, x_r)N = 0$*. Then*

$$\operatorname{Ext}_R^r(N, M) \cong \operatorname{Hom}(N, M/(x_1, x_2, \ldots, x_r)M)$$

Proof. We prove by induction on r. Since x_1 is non zero divisor of M, then we have the exact sequence

$$0 \to M \xrightarrow{\cdot x_1} M \to M/x_1M \to 0.$$

This yields the exact sequence

$$\operatorname{Ext}_R^{r-1}(N, M/x_1M) \to \operatorname{Ext}_R^r(N, M) \xrightarrow{\cdot x_1} \operatorname{Ext}_R^r(N, M). \tag{1}$$

Since $x_1N = 0$, then the last map in (1) is zero. When $r = 1$, it suffices to show that $\operatorname{Hom}(N, M) = 0$. Notice that since $x_1N = 0$, then $x_1f(n) = f(x_1n) = 0$ for all $n \in N$ and $f \in \operatorname{Hom}(N, M)$. Since x_1 is not a zero-divisor of M, then $f(n) = 0$ for all $n \in N$ and $f \in \operatorname{Hom}(N, M)$. Therefore $\operatorname{Hom}(N, M) = 0$. In particular, $\operatorname{Hom}(N, M/(x_1, x_2, \ldots, x_{r-1})M) = 0$ since $x_rN = 0$ and x_r is not a zero divisor of $(x_1, x_2, \ldots, x_{r-1})M$.

We now assume that $\operatorname{Ext}_R^{r-1}(N, M) = \operatorname{Hom}(N, M/(x_1, x_2, \ldots, x_{r-1})M)$. Therefore $\operatorname{Ext}_R^{r-1}(N, M) = 0$. From the exact sequence (1), we get

$$\operatorname{Ext}_R^r(N, M) = \operatorname{Ext}^{r-1}(N, M/x_1M).$$

Notice that by Theorem 2.9 the sequence $(x_2, \ldots, x_r)$ is a regular sequence of M/x_1M. Since $(x_2, \ldots, x_r)N = 0$, apply the induction to M/x_1M, we see that

$$\operatorname{Ext}_R^{r-1}(N, M/x_1M) \cong \operatorname{Ext}_R^r(N, M/(x_1, x_2, \ldots, x_r)M).$$

□

From the proof, we also get the following corollary,

Corollary 2.17. *The depth of an* R*-module* M *in an ideal* I *is*

$$\operatorname{depth}_I M = \min\{r \mid \operatorname{Ext}_R^r(R/I, M) \neq 0\}.$$

Now we are ready to give the first definition of Cohen-Macaulayness.

Definition 2.18. A Noetherian ring R is ***Cohen-Macaulay*** if for every ideal $I \neq R$, height $I = \operatorname{depth}_I R$.

Lemma 2.19. *Let* M *be a finitely presented* R*-module and* N *be any* R*-module, then*

$$\operatorname{Hom}(M, N)_{\mathfrak{p}} \cong \operatorname{Hom}(M_{\mathfrak{p}}, N_{\mathfrak{p}}),$$

for any prime ideal $\mathfrak{p}$.

Theorem 2.20. *Let R be a Noetherian ring. Then the following are equivalent:*

(1) *R is a Cohen-Macaulay ring.*

(2) *Localizations R_p at all prime ideals are Cohen-Macaulay rings.*

(3) *Localizations R_m at all maximal ideals are Cohen-Macaulay rings.*

Proof. The equivalence follows from Theorem 2.16, Lemma 2.19 and the fact that $M = 0$ if and only if $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$. □

Corollary 2.21. *Let R be a Noetherian ring. Then the following are equivalent:*

(1) height I = depth$_I$ R *for every proper ideal I.*

(2) height $\mathfrak{p}$ = depth$_{\mathfrak{p}}$ R *for every prime ideal $\mathfrak{p}$.*

(3) height $\mathfrak{m}$ = depth$_{\mathfrak{m}}$ R *for every maximal ideal $\mathfrak{m}$.*

Geometrically, x is not a zero divisor of M means that x does not vanish on any component of supp(M). Then a M-sequence will cut out supp(M) properly. Therefore, depth provides way to count the dimension. This ideal was used by Serre to generalize Cohen-Macaulayness to Serre's conditions.

Definition 2.22 (Serre's conditions)**.** An R-module M is said to satisfy condition (S_n) if

$$\operatorname{depth}_{\mathfrak{p}} M \geq \min\{\dim_{\mathfrak{p}} M, n\}, \quad \forall \mathfrak{p} \in \operatorname{Spec}(R).$$

Roughly speaking, Serre's condition (S_n) means that you can cut supp$M$ properly at most n times. A more detailed geometric interpretation is as follows,

(S_1) means the existence of non-zero-divisors, equivalently no embedded components

(S_2) means S_1 and being saturated in codimension 2, equivalently, functions defined outside a codimension 2 subvariety have unique extensions.

...

(S_n) means that every ("true") hypersurface has the (S_{n-1}) property.

Definition 2.23. Let M be a finitely generated module over a graded ring R. Let $\mathcal{F}$ be the coherent sheaf on Proj$R$ associated to M. We say that M is saturated if the natural R-module morphism

$$\varphi : M \to \Gamma_*(\mathcal{F}) := \bigoplus_{n\in\mathbb{Z}} \Gamma(\operatorname{Proj} R, \mathcal{F}(n))$$

is an isomorphism.

A R-module M is Cohen-Macaulay, if and only if M satisfies (S_i), for all $i = 1, 2, \ldots, \dim M$.

Example 2.24. All 0-dimension Noetherian rings are Cohen-Macaulay rings.

Example 2.25. All 1-dimension Noetherian local rings without nilpotent are Cohen-Macaulay rings. For example, $k[x, y]/(y^2)$ is CM, but $k[x, y]/(x^2, xy)$ is not.

Example 2.26. The ring $k[x, y, z, w]/(xw - yz)$ is Cohen-Macaulay.

Example 2.27. The ring $k[x^4, x^3y, xy^3, y^4]$ is not Cohen-Macaulay.

Definition 2.28 (Embedded Primes)**.** A prime ideal $\mathfrak{p} \in \text{Ass}(M)$ is called a embedded prime is $\mathfrak{p}$ is not a minimal prime.

Example 2.29. Consider $k[x,y]/(x^2, xy)$. The maximal ideal is a embed prime. Since $(x) \subset (x,y)$ and (x) is prime.

Proposition 2.30. *A Noetherian ring R is Cohen-Macaulay if and only if the completion $\hat{R}$ is Cohen-Macaulay.*

Proposition 2.31. *A Noetherian ring R is Cohen-Macaulay if and only if the polynomial ring $R[x]$ is Cohen-Macaulay.*

3. Cohen-Macaulay Rings: Characterizations

To understand, what does "Cohen-Macaulay" really mean, in this section, we review some consequence of the definition, alternative characterizations, and examples.

Proposition 3.1. *Any local C-M ring is equidimensional.*

Geometrically, this says that if a variety X is locally C-M at a point p, then p can not lie on two components of different dimensions.

Theorem 3.2 (Hartshorne's Connectedness Theorem)**.** *Let R be a local ring and let I and J be proper ideals of R whose radicals are incomparable. If $I \cap J$ is nilpotent, then* $\text{depth}(I + J) \leq 1$*. In particular, if R is C-M, or even satisfies Serre's condition (S_2), then* $\text{codim}(I + J) \leq 1$.

Geometrically, this means that at a C-M point, a variety must be locally "connected in codimention 1" in the sense that removing a subvariety of codmention 2 or more cannot disconnect it.

Most interseting Noetherian rings can be written as finitely generated modules over regular subrings. For example, Noether normalization allows us to write any affine algebra as a finitely generated module over a ploynomial ring of the same dimension.

Theorem 3.3. *Let R be a Noetherian regular local ring, or polynomial ring over a field k and S be a ring which is a finitely generated R-module, local in the first case, or graded in the second. If S is pure dimensional, in the sense that the localization of S at maximal ideals all have the same dimension, then S is Cohen-Macaulay if and only if S is a free R-module.*

Geometrically, this means that a variety X is Cohen-Macaulay if and only if for some finite morphism from X to a regular variety Y, the scheme-theoretic fibers of the morphism all have the same length.

Theorem 3.4 (Hochster-Roberts [HR74])**.** *Let X be a regular algebraic variety and G be a linearly reductive algebraic group acting on X. Then the geometry quotient $X//G$ of X is a Cohen Macaulay variety.*

The following theorem relates Cohen-Macaulayness with vanishing of cohomology groups.

Theorem 3.5 (Arithmetic Cohen-Macaulayness)**.** *Let X be a projective variety in $\mathbb{P}^n$ and R be the homogenous coordinate ring of X. Then R is Cohen Macaulay ring if and only if the following conditions hold:*

(1) *The natural morphism* $R \to \Gamma_*(\mathcal{O}_X) := \oplus_{k\in\mathbb{Z}} H^0(X, \mathcal{O}_X(k))$ *is bijective,*

(2) $H^i(X, \mathcal{O}_X(k)) = 0$ *for all* $k \in \mathbb{Z}$ *and* $i = 1, 2, \ldots, \dim X - 1$.

Remark 3.6.

(1) There are two definition on arithmetic Cohen-Macaulayness.
   - The homogenous coordinate ring is Cohen-Macaulay.
   - The section ring is Cohen-Macaulay.

   This two definition are not equivalent (an example is given below). However, if the variety is normal and projectively normal, i.e. the coordinate ring is integrally closed, equivalently he natural morphism $R \to \Gamma_*(\mathcal{O}_X) := \oplus_{k\in\mathbb{Z}} H^0(X, \mathcal{O}_X(k))$ is bijective, then the two definition are the same.

(2) The natural morphism (1) is simply sending $x \in R_m$ to $H^0(\mathcal{O}_X(m))$. This is well-defined because x is nothing but a degree 1 global section of $\mathcal{O}_X(m) = \widetilde{R(m)}$.

(3) Although $\widetilde{R}$ and $\widetilde{\Gamma_*(\mathcal{O}_X)}$ are both isomorphic to $\mathcal{O}_X$, the isomorphism in (1) in general is neither injective nor surjective. This is because the exact functor $\widetilde{\cdot}$ kinds higher order graded terms. More precisely, for a finitely generated R-module M, the associated coherent sheaf $\widetilde{M} = 0$ if and only if the n-th graded terms $M_n = 0$ for all sufficiently large n.

(4)
   - If R is normal, then the first condition is automatic satisfied. In fact, $\Gamma_*(\mathcal{O}_X)$ is the integral closure of R if X is normal and integral (see Exercise 5.14, Chapter II, [Har77]).
   - A projective variety X is called arithmetically normal if $\Gamma_*(\mathcal{O}_X)$ is the integral closure of R.
   - Consider the embedding $\mathbb{P}^1 \to \mathbb{P}^4$ which sends $[s, t]$ to $[t^4, t^3s, ts^3, s^4]$. The homogenous coordinate ring is $R = k[t^4, t^3s, ts^3, s^4]$, while the section ring is $k[t^4, t^3s, t^2s^2, ts^3, s^4]$. Moreover, the coordinate ring is not Cohen-Macaulay, while the section ring is Cohen-Macaulay and is the integral closure of R.
   - Note that normality and connectedness implies irreducibility. Because, otherwise we can find two non-zero elements $f, g \in \mathcal{O}_{X,x}$ but $fg = 0$ at x, where is a point in the intersection of irreducible components.

(5) If X is UFD, then $\mathrm{Pic}(X) = \mathbb{Z}$. Hence, if R is UFD, then R is Cohen-Macaulay if and only if $H^i(X, L) = 0$ for every invertible sheaf and $i = 1, 2, \ldots, \dim X - 1$. Consequently, Cohomology groups of intermediate dimension of all invertible sheaves on Grassmannian varieties vanish.

Our proof is an imitation of the proof of Proposition 1.2 in [Bea00].

Proof. Let CX be the affine cone of X with vertex 0. Then $X = (CX \setminus \{0\})/\mathbb{C}^*$. Let $p : U := CX \setminus \{0\}) \to X$ be the natural quotient. Then $p_*\mathcal{O}_U = \oplus_{k\in\mathbb{Z}} \mathcal{O}_X(k)$. (Note that the characters of $\mathbb{C}^*$ are $\mathbb{Z}$. the eigen-sheaves of $p_*\mathcal{O}_U$ are line bundles on X. Moreover, since p is the restriction of the morphism $p : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Then the line bundles are exactly $\mathcal{O}_X(k)$.) Therefore, $H^i(U, \mathcal{O}_U) = \oplus_{k\in\mathbb{Z}} H^i(X, \mathcal{O}_X(k))$. Consider the long exact sequence of local cohomology

$$\cdots \to H^i_{\{0\}}(CX, \widetilde{\mathcal{F}}) \to H^i(CX, \widetilde{\mathcal{F}}) \to H^i(U, \widetilde{\mathcal{F}}) \to \cdots,$$

where $\widetilde{\mathcal{F}}$ is the extension of $\mathcal{F}$ and $\mathcal{F} = \mathcal{O}_U$. Notice that $H^0(CX, \widetilde{\mathcal{F}}) \to H^0(U, \widetilde{\mathcal{F}})$ and $H^i(CX, \widetilde{\mathcal{F}}) = 0$ for $i > 0$. Then $H^0_{\{0\}}(CX, \widetilde{\mathcal{F}}) = H^1_{\{0\}}(CX, \widetilde{\mathcal{F}}) = 0$ and $H^i(U, \widetilde{\mathcal{F}}) \cong H^{i+1}_{\{0\}}(CX, \widetilde{\mathcal{F}})$. Therefore, $\widetilde{\mathcal{F}}_0$ is Cohen-Macaulay if and only if $H^i(U, \widetilde{\mathcal{F}}) = 0$ for all $i = 1, 2, \ldots, \dim X - 1$. Since p is smooth. Then $\widetilde{\mathcal{F}}_v$ being Cohen-Macaulay for all $v \in U$ is equivalent to that $\mathcal{O}_X$ being Cohen-Macaulay. The theorem then follows from the following proposition. □

Proposition 3.7. *Let X be a projective variety. Then X is Cohen-Macaulay if and only if $H^p(X, \mathcal{O}_X(-k)) = 0$ for $0 < p < \dim X$ and $k >> 1$.*

A quick proof involves dialyzing complexes. The following is a more elementary proof.

Proof. Let $i : X \to \mathbb{P}^N$ be the embedding. We shall still denote by $\mathcal{O}_X(-k)$ the zero extension $i_!(\mathcal{O}_X(-k))$ on $\mathbb{P}^N$. Notice that $i_! = i_*$ and $R^p i_* = 0$ for $p \geq 1$. Therefore, we have $H^p(X, \mathcal{O}_X(-k)) = H^p(\mathbb{P}^N, \mathcal{O}_X(-k))$ Apply Serre duality to $\mathbb{P}^N$, we get

$$H^p(\mathbb{P}^N, \mathcal{O}_X(-k)) = \mathrm{Ext}^{N-p}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X(-k), \omega_{\mathbb{P}^N}) = H^0(\mathbb{P}^n, \mathcal{E}xt^{N-p}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X(-k), \omega_{\mathbb{P}^N})).$$

Use the adjointness of Hom and $\otimes$, we see that

$$H^0(\mathbb{P}^n, \mathcal{E}xt^{N-p}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X(-k), \omega_{\mathbb{P}^N})) = H^0(\mathbb{P}^n, \mathcal{E}xt^{N-p}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega_{\mathbb{P}^N}) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^N}(k)).$$

Therefore,

$$H^p(X, \mathcal{O}_X(-k)) = 0 \text{ for } 0 < p < \dim X \text{ and } k >> 1$$

if and only if

$$\mathcal{E}xt^{N-p}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega_{\mathbb{P}^N}) = 0, \text{ for } 0 < p < \dim X$$

if and only if

$$\mathcal{E}xt^{N-p}_{\mathcal{O}_{\mathbb{P}^N,x}}(\mathcal{O}_{X,x}, \omega_{\mathbb{P}^N,x}) = 0 \text{ for } p < \dim X \text{ and } x \in X.$$

By local duality,

$$\mathcal{E}xt^{N-p}_{\mathcal{O}_{\mathbb{P}^N,x}}(\mathcal{O}_{X,x}, \omega_{\mathbb{P}^N,x}) = H^i_{\mathfrak{n}}(\mathbb{P}^N, \mathcal{O}_{X,x}).$$

Therefore,

$$H^p(X, \mathcal{O}_X(-k)) = 0 \text{ for } 0 < p < \dim X \text{ and } k >> 1$$

if and only if

$$H^i_{\mathfrak{n}}(\mathbb{P}^N, \mathcal{O}_{X,x}) = 0 \text{ for } i < n \text{ and } x \in X, \text{ where } \mathfrak{n} \text{ is the maximal ideal of } \mathcal{O}_{P,x}.$$

But notice that $H^i_{\mathfrak{n}}(\mathbb{P}^N, \mathcal{O}_{X,x}) = H^i_{\mathfrak{m}}(X, \mathcal{O}_{X,x})$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,x}$. Therefore, X is Cohen-Macaulay if and only if $H^p(X, \mathcal{O}_X(-k)) = 0$ for $0 < p < \dim X$ and $k >> 1$.

□

Theorem 3.8 (Unimixedness Theorem)**.** *Let R be a Cohen-Macaulay ring. Then ideals generated by R-sequences are unimixed, i.e. all associated prime ideals are minimal.*

Intersection multiplicity. There are two essential ways to define intersection multiplicity of two varieties. Consider the intersection of $y = x^2$ and $y = 0$ in $\mathbb{A}^2$. The first (topological) method is based on intuition on coherence of deformation. If we shift $y = 0$ slightly to $y = c$ with $c \neq 0$. Then we get two distance points. So it is naturally enough to regard 2 as the intersection multiplicity of the two curves. A different points view (scheme-theoretical) is to define the intersection multiplicity as $l((k[x,y]/(y-x^2,y))_{\mathfrak{m}})$, where $\mathfrak{m}$ is the maximal ideal of the origin and l is the length function, i.e. the length of filtration such that all factors are isomorphic to the residue field k. Both ideals can be generalized easily to higher dimensions, but they do not always give the same answer. In fact, the multiplicities computed in these two way agree if and only if both varieties are Cohen-Macaulay.

Definition 3.9. Let X be a regular algebraic variety and Y and Z be Cohen-Macaulay subvarieties over a algebraic closed field k. Assume x is an isolated singularity of $Y \cap Z$, then the intersection multiplicity of Y and$Z$ at x is $i_x(Y,Z) = l((R/(I+J))_{\mathfrak{m}}$, where R is the coordinate ring of X, I and J are the ideals of Y and Z in Z, and $\mathfrak{m}$ is the maximal ideal at x.

Remark 3.10. For arbitrary algebraic varieties $X = \mathrm{Spec} R/I$ and $Y = \mathrm{Spec} R/J$, Serre "corrects" the definition by the Euler characteristic of Tor:

$$i_x(X,Y) = \sum_j (-1)^j l(\mathrm{Tor}^j(R/I, R/J)).$$

Depth and local cohomology. We first recall the definition of local cohomology modules.

Definition 3.11. Let I be an ideal in R and M be an R-module. We define the 0-th cohomology module of M with support I to be

$$H_I^0(M) = \lim_{n\to\infty} \mathrm{Hom}(R/I^n, M).$$

The i-th cohomology module of M with support I is

$$H_I^i(M) \cong \lim_{n\to\infty} \mathrm{Ext}^i(R/I^n, M).$$

If we view M as $\Gamma(\mathrm{Spec} R, \tilde{M})$ the global section of the associated sheaf $\tilde{M}$, then $H_I^0(M)$ are just sections with support on the closed subscheme $\mathrm{Spec} R/I \subset \mathrm{Spec} R$.

Theorem 3.12. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. Then*

(1) $H_{\mathfrak{m}}^i(M) = 0$ *for* $i < \mathrm{depth}\ M$ *and for* $i > \dim\ M$*, and*

(2) $H_{\mathfrak{m}}^i(M) \neq 0$ *for* $i = \mathrm{depth}\ M$ *and for* $i = \dim\ M$.

Proof. This is a consequence of Corollary 2.14 and 2.17 . □

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