

# Filtrations of Vector Bundles on Smooth Projective Algebraic Curves

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The bundles we will talk about are all algebraic vector bundles on a smooth projective algebraic curve.

**Definition 0.1.** The slope of a vector bundle  $E$ , denoted by  $\mu(E)$ , is defined to be  $\frac{\deg E}{\operatorname{rk} E}$ .  $E$  is semi-stable (stable) if  $\mu(F) \leq \mu(E)$  (resp.  $\mu(F) < \mu(E)$ ) for any sub-bundle  $F \subset E$ . Equivalently,  $\mu(E/F) \geq \mu(E)$  (resp.  $\mu(E/F) > \mu(E)$ ).

Let  $X$  be a smooth projective algebraic curve. Given a rational number  $\mu$ , we define  $\mathcal{C}(\mu) = \{E \text{ semi-stable bundle of slope } \mu \text{ on } X\}$

**Proposition 0.2.**  $\mathcal{C}(\mu)$  is an abelian sub-category of category of vector bundles.

*Proof.* We only need to show that the following two properties hold.

(1) Additivity:

It is easy to show that  $\mu(E \oplus F) = \mu$  if  $\mu(E) = \mu(F) = \mu$ . The semi-stability follows from the following fact:

**Lemma 0.3.** Let  $E, F \in \mathcal{C}(\mu)$  and  $0 \rightarrow E \rightarrow G \rightarrow F \rightarrow 0$  exact. Then  $G$  is semi-stable and  $\mu(G) = \mu$ .

*Proof.* Clearly,  $\mu(G) = \mu$ . Take a sub-bundle  $G'$  of  $G$ , we will get a short exact sequence  $0 \rightarrow E' = E \cap G' \rightarrow G' \rightarrow F' \rightarrow 0$  with  $E' \subset E$  and  $F' \subset F$ . Hence,  $\mu(G') \leq \mu(F') = \mu(F) = \mu = \mu(G)$ . Hence  $G$  is semi-stable.  $\square$

(2) Kernel and cokernel:

Let  $E, F \in \mathcal{C}(\mu)$  and  $f : E \rightarrow F$  is a bundle map. Then we have the following short exact sequences:  $0 \rightarrow \ker f \rightarrow E \rightarrow \operatorname{im} f \rightarrow 0$  and  $0 \rightarrow \operatorname{im} f \rightarrow F \rightarrow \operatorname{coker} f \rightarrow 0$

By definition,  $\mu = \mu(E) \leq \mu(\operatorname{im} f) \leq \mu(F) = \mu$ . Hence  $\mu(\operatorname{im} f) = \mu$ , so that  $\mu(\ker f) = \mu$ . This implies that  $\ker(f)$  is semi-stable, since any subbundle of  $\ker f$  is also a subbundle of  $E$ .

Similarly,  $\mu(\operatorname{coker} f) = \mu$ .  $\operatorname{coker} f$  is a vector bundle follows from the following result:

**Lemma 0.4.** *Let  $\mu(E) = \mu(F)$  and  $E \subset F$ . Then  $F/E$  is torsion free, hence a vector bundle*

*Proof.* It is clear that  $E/F$  can be written as  $E/F = Q \oplus T$ , where  $Q$  is a vector bundle and  $T$  is the torsion subsheaf of  $E/F$ . Then the following diagram

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 & & & & & & T \\
 & & 0 & \longrightarrow & 0 & \longrightarrow & \downarrow \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & E/F \longrightarrow 0 \\
 & & \downarrow f & & \downarrow & & \downarrow \\
 0 & \longrightarrow & G & \longrightarrow & F & \longrightarrow & Q \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \operatorname{coker} f & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & & 
 \end{array}$$

commutes, in which the rows and columns are all exact. Hence by snake lemma, we know  $T = \operatorname{coker} f$ . If  $T \neq 0$ , then  $\deg T > 0$  and  $\operatorname{rk} T = 0$ . Therefore  $\mu = \mu(F) \geq \mu(G) > \mu(E) = \mu$ . Contradiction!  $\square$

Note that any sub-bundle  $S$  of  $\operatorname{coker} f$  can be realized as a quotient  $B/\operatorname{im} f$  of a

subbundle  $B$  of  $F$  by  $\text{im } f$ . If  $\mu(S) > \mu$  then  $\mu(B)$  must be strictly greater than  $\mu$ . A contradiction! Therefore  $\text{coker } f$  is semi-stable.  $\square$

It is easy to see, in  $\mathcal{C}(\mu)$ , a descending filtration will be eventually stationary, because the rank decrease.

**Lemma 0.5.** *Let  $E \in \mathcal{C}(\mu)$ . If  $E$  is not stable, then  $E$  has a stable sub-bundle  $F$  of slope  $\mu$ .*

*Proof.* pick up a any sub-bundle  $E_1 \in \mathcal{C}(\mu)$  of  $E$ . If  $E_1$  is stable, then we are done. If not we can pick up a sub-bundle  $E_2 \in \mathcal{C}(\mu)$  of  $E_1$ . If  $E_2$  is stable, then we are done. If  $E$  has no stable sub-bundle at all, keep searching in this way, we will eventually get an descending chain  $E \supset E_1 \supset E_2 \supset \dots \supset \dots$  of infinite length. But the  $\text{rk } E$  is fixed. So we must have a stable sub-bundle.  $\square$

**Theorem 0.6** (Jordan-Holder Filtration). *Let  $F$  be a semi-stable bundle of slope  $\mu$ . Then there exists a filtration, called Jordan-Holder filtration  $0 \subset F_1 \subset F_2 \subset \dots \subset F_k = F$  such that  $gr_i := F_i/F_{i-1}$  is stable. If  $0 \subset F'_1 \subset F'_2 \subset \dots \subset F'_{k'} = F$  is another Jordan-Holder filtration, then  $k = k'$  and  $\{gr_i\} = \{gr'_i\}$ .*

*Proof.* We will only discuss the existence. By the above lemma, there exist a  $F_1 \subset F$  such that  $F_1$  is stable of slope  $\mu$ , hence  $F/F_1$  is semi-stable of slope  $\mu$ . Then there exists a sub-bundle  $F \supset F_2 \supset F_1$ , such that  $F_2/F_1$  is stable in  $F/F_1$ . Repeat this procedure, since the rank of  $F$  is finite, we will eventually get the filtration.  $\square$

Jordan-Holder filtration allows us to deal with semi-stable bundles in terms of stable bundle. The next filtration will help us classify un-stable bundles in terms of semi-stable bundle.

**Lemma 0.7.** *Let  $E$  be a bundle on  $X$ . Then  $\text{deg } F$  is bounded function over  $\{F | F \subset E\}$ .*

*Proof.* If  $X = \mathbf{P}^1$ , then  $E = \bigoplus_{i=1}^{\text{rk } E} \mathcal{O}(a_i)$  and  $F = \bigoplus_{j=1}^{\text{rk } E} \mathcal{O}(b_j)$ . Since  $F$  is a sub-bundle of  $E$  then  $\max\{b_j\} \leq \max\{a_i\}$ . So  $\text{deg } F \leq \text{rk } E \cdot \max\{a_i\}$ . Assume that  $g(X) \geq 1$ . It is well know that there exists a finite morphism  $f : X \rightarrow \mathbf{P}^1$ . Note that in this case  $f$  is flat thanks to the following theorem:

**Theorem 0.8** (Hartshorne Proposition III.9.7). *Let  $X$  and  $Y$  be two smooth projective variety and  $\dim Y = 1$ . If  $f : X \rightarrow Y$  is finite covering then  $f$  is flat.*

**Proposition 0.9** (see Eisenbud). *A coherent flat sheaf is locally free.*

Therefore  $f_*F$  is a sub-bundle of the vector bundle  $f_*E$ . Notice that  $\chi(F) = \chi(f_*F)$ , since  $f$  is finite. By Riemann-Roch,  $\deg F$  is bounded because  $\deg f_*F$  is bounded.  $\square$

**Theorem 0.10.** *Let  $E$  be a vector bundle on  $X$ . Then there is a filtration, called Harder-Narasimhan filtration,  $0 \subset E_1 \subset E_2 \subset \dots \subset E_k = E$ , such that the grading  $gr_i := E_i/E_{i-1}$  satisfies:*

1.  $gr_i$  is semi-stable,
2.  $\mu(gr_i) > \mu(gr_{i+1})$ .

*Moreover the filtration is unique.*

*Proof.* We will only show the existence. If  $E$  is already semi-stable, we are done. If not, let  $E_1$  be the sub-bundle of  $E$  with maximal slope and maximal rank among the sub-bundles of  $E$ . Clearly,  $E_1 \neq 0$ . We claim that  $E_1$  is semi-stable. Let  $G$  be any sub-bundle of  $E$ . By the choice of  $E_1$ ,  $\mu(G) \leq \mu(E_1)$ . Therefore  $E_1$  is semi-stable. If  $E/E_1$  is semi-stable, we are done. Otherwise, there exists a non-zero semi-stable sub-bundle  $E_2/E_1$  of  $E/E_1$ . We will show that  $\mu(E_2/E_1) < \mu(E_1)$ . Otherwise, that  $\mu(E_2/E_1) \geq \mu(E_1)$  implies that  $\mu(E_2) \geq \mu(E_1)$  and  $rk E_2 > rk E_1$ , however,  $E_1$  is the maximal element among the sub-bundles, therefore  $E_2 = E_1$ . Contradiction! By repeating this procedure, we will get the finite filtration.  $\square$