Invariant Geometry

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April 6, 2010

Throughout this notes, the ground field will be $\mathbf{C}$.

1 Algebraic Group Actions and Quotients

1.1 Actions

Let G be an affine algebraic group and Y be an algebraic variety.

Definition 1.1. An action of G on Y is a morphism $\sigma : G \times Y \to Y$, such that

1. $\forall g \in G, \sigma_g = \sigma|_{g \times Y} \in Aut(Y)$;
2. $G \to Aut(Y)$ given by $g \mapsto \sigma_g$ is a group morphism.

We usually denote by $g \cdot y \ (= \sigma(g, y))$ the action.

For any $y \in Y$, $O^G(y) := \{g \cdot y | g \in G\}$ is called the orbit of y. $S^G(y) := \{g \in G | gy = y\}$ is called the stablizer of y.

Example 1.2. Let $G = \mathbf{C}^*$ and $Y = \mathbf{C}^2$. The following is an action:

$$\begin{aligned} \sigma : \mathbf{C}^* \times \mathbf{C}^2 &\to \mathbf{C}^2 \\ (\lambda, (x, y)) &\mapsto (\lambda x, \lambda^{-1} y). \end{aligned}$$

We have the following types of orbits:

- $O^G((x, y)) = \{(x', y')|x'y' = xy\}$, for $xy \neq 0$;
- $O^G((x, 0)) = \{(\lambda x, 0)\}$, for $x \neq 0$;
- $O^G((0, y)) = \{(0, \lambda^{-1} y)\}$, for $y \neq 0$;
- $O^G((0, 0)) = \{(0, 0)\}$.

Note that $O^G((x, 0))$ and $O^G((0, y))$ are note closed, moreover

$$\overline{O^G((x, 0))} \cap \overline{O^G((0, y))} = O^G((0, 0)).$$

Definition 1.3. A morphism $\varphi : X \to Y$ is a G-morphism if the following diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{Id \times \varphi} & G \times Y \\ \downarrow \sigma & & \downarrow \sigma \\ X & \xrightarrow{\varphi} & Y \end{array}$$

commutes. We say φ is G-invariant, if $\varphi(O^G(x)) = y$ for any $x \in X$.

In other words, a G-invariant morphism is a G-morphism which is constant on all orbits.

1.2 Categorical quotients

Given an action, what is a quotient of the action, or what properties should a quotient satisfy. The reason that we study quotient is that we like the quotient to parametrize all orbits of the action. A property of a quotient is that the quotient morphism must be a G-invariant morphism. So to get a quotient, we may search it in the set of G-invariant morphisms. However, in general, the G-invariant morphisms may not be able to parametrize all orbits of the action. That means, a "good" quotient may not exist. However, if it can provide finest possible parametrization of the orbits, it still be useful. Under this philosophy, Here it comes the definition of categorical quotient:

Definition 1.4 (Categorical Quotient)**.** Assume that G acts on Y. A pair (Z, φ) consisting of a scheme Z and a morphism $\varphi : Y \to Z$ is called a categorical quotient of Y by G, if it satisfies the following universal property, i.e. given any G-invariant morphism

$\varphi' : Y \to Z'$, there exist a unique morphism $\gamma : Z \to Z'$ such that the following diagram

$$\begin{array}{ccc} Y' & & \\ {\scriptstyle\varphi}\big\downarrow & \overset{\varphi}{\searrow} & \\ Z & \xrightarrow{\gamma} & Z' \end{array}$$

commutes.

Example 1.5. Consider the action given in Example 1.2. Then there is a categorical quotient given by $\mathbf{C}^2 \overset{\pi}{\to} \mathbf{C}$ such that $\pi(x, y) = xy$. It is easy to see that $\pi^{-1}(c) = \{(x, y)|xy = c\} = O^G(c)$ if $c \neq 0$, while $\pi^{-1}(0) = O^G((x, 0)) \cap O^G((0, y)) \cap O^G((0, 0))$. So $\mathbf{C}$ parametrizes only closed orbits.

1.3 Good quotients

In some sense, to study the varieties is equivalent to study the regular functions on it. A function is constant on orbits if and only if it is G-invariant. So given a variety Y with an action G, it is naturally to consider the varieties X with property that $A(X) = A(Y)^G$, where $A(X)$ is the coordinate ring.

A finner quotient is involving the invariant subring.

Definition 1.6. Let G be an affine algebraic group and Y be an algebraic variety. We say that a pair (Z, φ) consisting of an algebraic variety Z and a morphism $\varphi : Y \to Z$ is a good quotient is if it satisfies the following properties:

1. φ is G-invariant.
2. φ is surjective and affine such that for each affine open subset $U \subset Z$, $A(U) = A(\varphi^{-1}(U)^G$.
3. The images of G-invariant closed subsets are closed.
4. Any two disjoint G-invariant closed subsets have disjoint images.

Example 1.7. Let $\pi : \mathbf{C}^2 \to \mathbf{C}$ be a map such that $\pi(x, y) = xy$. One can check easily that $(\mathbf{C}, \varphi)$ is a good quotient.

Also, note that $\mathbf{C}[x, y] \supset \mathbf{C}[x, y]^G = \mathbf{C}[xy] \overset{t=xy}{\cong} \mathbf{C}[t]$. Then π is exactly the induced morphism $\mathbf{C}^2 = \mathrm{Spec}\mathbf{C}[x, y] \to \mathrm{Spec}\mathbf{C}[x, y]^G = \mathbf{C}$. We will see later, a morphism formed in this way is indeed a good quotient.

Definition 1.8. A good quotient is called a geometric quotient if each orbit is closed.

Example 1.9. Let $\mathbf{C}^*$ act on $\mathbf{C}^{n+1} \setminus \{0\}$ by multiplication. Then $\varphi : \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{CP}^n$ is a geometric quotient.

2 Existence of geometric quotients

Let G be an algebraic group and V be a vector space. A representation of G on V is a homomorphism $\rho : G \to GL(V)$. V can be viewed as a G-module in the sense that, $g \cdot v = \rho(g)v$. A subspace $W \subset V$ is called a G-submodule, if it is G-invariant, i.e. $G \cdot W \subset W$.

Definition 2.1. ρ is irreducible if V is a simple G-module. ρ is totally reducible if it decomposes in a direct sum of irreducible representations, equivalently, V is a semi-simple G-module, i.e. V is the direct sum of simple G-modules. G is linearly reductive if all the G-module V are semi-simple.

Lemma 2.2. *G is linearly reducible if and only if for any finite dimensional G-module V and G-submodule W, there exists a G-submodule W' such that $V = W \oplus W'$.*

Lemma 2.3. *Let G be an affine algebraic group. Suppose that G contains a compact subgroup K (in the analytic topology) such that K is Zariski-dense in G. Then G is linearly reductive.*

Example 2.4. $GL(n)$, $PGL(n)$, $SL(n)$ are all linearly reductive. Because they contain Zariski-dense compact subgroup $U(n)$, $SU(n)$ and $PU(n)$ respectively.

2.1 Affine case

Theorem 2.5 (Hilbert-Mumford)**.** *Let G be a linearly reductive algebraic group acting on the affine algebraic variety $Y = \mathrm{Spec}A$. Then the $\mathbf{C}$-algebra A^G is of finite type. Let $Z = \mathrm{Spec}A^G$ and $\varphi : Y \to Z$ be the morphism induced by the inclusion $A^G \hookrightarrow A$. Then (Z, φ) is a good quotient.*

2.2 Projective case

Let $Y \subset \mathbf{CP}^n$ and $A(Y)$ be its polynomial ring. Let G be an affine algebraic group acting on Y. To construct a good quotient of Y, it is natural to consider $\mathrm{Proj} A^G$. However, in general, the induced map $Y = \mathrm{Proj} A(Y) \to \mathrm{Proj} A(Y)^G$ may not well-defined.

Example 2.6. Consider the $\mathbf{C}*$ action on $\mathbf{CP}^2$ given by $\lambda \cdot [x, y, z] = [\lambda x, \lambda^{-1} y, z]$. Then $\mathbf{C}[x, y, z]^G = \mathbf{C}[xy, z] \cong \mathbf{C}[t, z]$. $\mathrm{Proj}\mathbf{C}[t, z] = \mathbf{CP}^1$. The induced map $[x, y, z] \mapsto [xy, z]$ is not well-defined at $[1, 0, 0]$ and $[0, 1, 0]$. So we have to remove some points. Note that all the G-invariant non-trivial polynomials are vanish at those two points. Those are the unstable points in the following sense.

Definition 2.7. A point $y \in Y$ is said to be semi-stable under the action of G, if there exist a homogeneous G-invariant polynomial f such that $\deg f > 0$ and $f(y) \neq 0$. Otherwise, y is called unstable.

Let Y^{ss} be the open subset of semi-stable points of Y under the action of G and $Z = \mathrm{Proj} A(Y)^G$. Then we have a morphism of algebraic varieties

$$\varphi : Y^{ss} \to Z.$$

Theorem 2.8. *The morphism $\varphi : Y^{ss} \to Z$ induced by $A(Y)^G \hookrightarrow A(Y)$ is a good quotient of the open set Y^{ss} of the semi-stable points of Y under the action of G.*

Definition 2.9. A point $y \in Y$ is said to be stable under the action of G if it is semi-stable and the orbit map $G \to Y^{ss}$ given by $g \mapsto g \cdot y$ is proper.

3 The Hilbert-Mumford criterion

3.1 Numerical criterion

Definition 3.1. A one parameter subgroup of an affine algebraic group G is an non-trivial morphism $\lambda : G_m = \mathbf{C}^* \to G$.

Definition 3.2. Let V be a G-module and $v \in V$. We say that v is

- semi-stable if $0 \notin \overline{O^G(v)}$;

- stable if the orbit $O^G(v)$ is closed and the stabilizer $S^G(v)$ is finite.

Theorem 3.3. *Let G be a linearly reductive group and V be a G-module. A point $y \in V$ is semi-stable (stable) under the action of G if and only if for all one parameter subgroups λ it is semi-stable (respectively stable) under the action of $\mathbf{C}^*$ given by λ.*

This theorem gives easily a numerical criterion for semi-stability.

Given a one parameter subgroup $\lambda : \mathbf{C}^* \to G$ acting on $\mathbf{C}^{n+1}$. Then λ can be diagonalized. Let $e_0, \cdots, e_n$ be the basis diagonalizes λ. Then for each $t \in \mathbf{C}^*$, $t \cdot e_i := \lambda(t) \cdot e_i = t^{r_i} e_i$ for some integer r_i. Let $x \in \mathbf{CP}^n$ which is represented by $\hat{x} := \sum_{i=0}^{n} x_i e_i \in \mathbf{C}^{n+1} \setminus \{0\}$. We define $\mu(\lambda, x) := \max\{-r_i | x_i \neq 0\}$.

Theorem 3.4. *Let G be a linearly reductive group acting on $\mathbf{C}^{n+1}$. Let $Y \subset \mathbf{CP}^n$ be a closed G-invariant subvariety. A point $y \in Y$ is semistable (stable) if $\mu(\lambda, y) \geq 0$ (respectively, $\mu(\lambda, y) > 0$) for all one parameter subgroups λ*

Remark 3.5. From the definition of $\mu(\lambda, y)$, we see that $\mu(\lambda, g \cdot y) = \mu(g^{-1}\lambda g, y)$. It allows us to replace λ by its conjugate. In particular, for $g \in SL(n+1)$, it is well know that every one parameter subgroup is conjugate to a diagonal matrix of the form

$$\begin{pmatrix} t^{r_0} & 0 & \cdots & 0 \\ 0 & t^{r_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{r_n} \end{pmatrix}$$

with $\sum r_i = 0, r_0 \geq r_1 \geq \cdots \geq r_n$ and not all $r_i = 0$.

3.2 Computations of semi-stable points

3.2.1 Binary forms

Proposition 3.6. *A point $(x_1, \cdots, x_d) \in \mathbf{CP}^{1(d)}$ is semi-stable under the induced action of $SL(2)$ on $\mathbf{CP}^1$ if no more than half of $x_1, \cdots, x_n$ coincide.*

Proof. Note that we can view $(x_1, \cdots, x_n)$ as the zero set of a binary form of degree n:

$$F_n(x, y) = \prod_{i=1}^{d}(b_i x - a_i y),$$

where $x_i = (a_i, b_i)$. In this view, $\mathbf{CP}^{1^{(d)}} = \mathbf{P}(\mathbf{C}[x, y]_d)$. The quotient we are consider is then $\mathbf{P}(\mathbf{C}[x, y]_d)//SL(2)$. Any one parameter subgroup λ of $SL(2)$ is conjugate to one of the form

$$\lambda_r(t) = \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}.$$

Let $\lambda_r(t)$ act on a degree d binary form $f := \sum_{i=0}^{d} a_i x^i y^{d-i}$ is given by

$$\lambda_r(t) \cdot f = \sum_{i=0}^{d} a_i t^{r(2i-d)} x^i y^{d-i}.$$

So $\mu(\lambda_r, f) = r(n - 2i_0)$, where i_0 is the smallest i such that $a_i \neq 0$. Thus, $\mu\lambda_r, f \geq 0$ if and only if $i_0 \leq [\frac{n}{2}]$. Equivalently, $a_i \neq 0$ for $i > [\frac{n}{2}]$, equivalent to no point of $\mathbf{CP}^1$ occurs as a point of multiplicity $[\frac{n}{2}]$. Therefore, $(x_1, \cdots, x_d) \in \mathbf{CP}^{1^{(d)}}$ is semi-stable if no more that half of them coincide. □

3.2.2 Grassmannian

Let H and V be two $\mathbf{C}$-vector spaces of dimension m and n respectively. Let $G(r, H \otimes V)$ be the Grassmannian of vector sub-spaces of dimension r in $H \otimes V$. The linear reductive group $SL(H)$ acts on $H \otimes V$ by $g \cdot (h \otimes v) = (g \cdot h) \otimes v$. It follows that $SL(H)$ acts on $G(r, H \otimes V)$ by $g \cdot k = (g \otimes Id)(k)$. We get a linear action of $SL(H)$ on $\wedge^r(H \otimes V)$ by $g \cdot (x_1 \wedge \cdots \wedge x_r) = g \cdot x_1 \wedge \cdots \wedge g \cdot x_r$. So an action on the projective space $\mathbf{P}(\wedge^r(H \otimes V))$ The Plucker embedding $G(r, H \otimes V) \hookrightarrow \mathbf{P}(\wedge^r(H \otimes V))$ is then $SL(H)$-equivariant, i.e. it commutes with the group action. As a closed invariant subvariety, we want to determine the semi-stable and stable points of $G(r, H \otimes V)$ under the $SL(H)$ action.

Proposition 3.7. *Let $K \in G(r, H \otimes V)$. Then the following are equivalent.*

1. *K is semi-stable (resp. stable) under $SL(H)$.*

2. *for each proper non-zero sub-space $H' \subset H$, let $K' = H' \otimes V \cap K$, we have*

$$\frac{\dim K'}{\dim H'} \leq \frac{\dim K}{\dim H} (\textit{ resp. } <).$$

Proof. Let λ be a one parameter subgroup of $SL(H)$, we need to compute $\mu(\lambda, K)$. From Remark 3.5, we can assume that

$$\lambda(t) = \begin{pmatrix} t^{r_0} & 0 & \cdots & 0 \\ 0 & t^{r_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{r_n} \end{pmatrix}$$

with $\sum r_i = 0$, $r_0 \geq r_1 \geq \cdots \geq r_n$ and not all $r_i = 0$. We can assume that there are only s different r_i, say $r_{i_1} > \cdots r_{i_s}$. Let $\oplus_{i=1}^{s} H_i$ be the irreducible decomposition of H associated to the action $\lambda(t)$. Then $K = \oplus[(H_i \otimes V) \cap K]$ and

$$\wedge^r (H \otimes V) = \oplus_{j_1 + \cdot j_s = r} \wedge^{j_1} (H_1 \otimes V) \otimes \cdots \otimes \wedge^{j_s} (H_s \otimes V).$$

Denote by $K_i = (H_i \otimes V) \cap K$. Then $\wedge^r K \cap \wedge^{j_1}(H_1 \otimes V) \otimes \cdots \otimes \wedge^{j_s}(H_s \otimes V) \neq 0$ if $j_i = \dim K_i$. Therefore

$$\begin{aligned} \lambda(t) \cdot K &= \max\{-\sum_{i=1}^{s} r_i j_i | \wedge^r K \cap \wedge^{j_1}(H_1 \otimes V) \otimes \cdots \otimes \wedge^{j_s}(H_s \otimes V) \neq 0\} \\ &= \sum_{i=1}^{s} r_i \dim K_i. \end{aligned}$$

Suppose that

$$\frac{\dim K'}{\dim H'} \leq \frac{\dim K}{\dim H}$$

for any $H' \subset H$ and $K' = H' \otimes V \cap K$. We see that for all $1 \leq t \leq s$,

$$\sum_{i=1}^{t} \dim K_i \leq \frac{\dim K}{\dim H} \sum_{i=1}^{t} \dim H_i.$$

Note that $\sum_{i=1}^{r} r_i \dim H_i = 0$. Then we have

$$\begin{aligned}
-\sum_{i=1}^{r} r_i \dim K_i &= -r_1 \dim K_1 + \sum_{t=2}^{s-1} r_t (\sum_{i=1}^{t} \dim K_i - \sum_{i=1}^{t-1} \dim K_i) \\
&= -\sum_{t=1}^{s-1} [(r_t - r_{t+1}) \sum_{i=1}^{t} \dim K_i + r_s r \\
&\geq -\frac{\dim K}{\dim H} \{ \sum_{t=1}^{s-1} [(r_t - r_{t+1}) \sum_{i=1}^{t} \dim H_i + r_s m \} \\
&= -\frac{\dim K}{\dim H} \sum_{i=1}^{r} r_i \dim K_i = 0.
\end{aligned}$$

Therefore K is semi-stable.

Conversely, let $H_1 \subset H$ be a subspace and H_2 its complement, i.e. $H_1 \oplus H_2 = H$. If K is semi-stable for all one parameter subgroups of $SL(H)$. It is semistable for the one parameter subgroup

$$\begin{pmatrix} t^{n-\dim H_1} I_{H_1} & 0 \\ 0 & t^{n-\dim H_2} I_{H_2} \end{pmatrix}$$

associated to the direct sum $H_1 \oplus H_2$, where I_{H_1} and I_{H_2} are the identity matrix over H_1 and H_2 respectively. Let $K_1 = H_1 \otimes V \cap K$. Then we have

$$\begin{aligned}
\mu(\lambda, K) &= -(n - \dim H_1) \dim K_1 - (n - \dim H_2) \dim K_2 \\
&= -(n - \dim H_1) \dim K_1 - (\dim H_1) \dim K_2 \\
&= m \dim H_1 - n \dim K_1 \geq 0.
\end{aligned}$$

Therefore, $m \dim H_1 \geq n \dim K_1$, equivalently, $\dfrac{\dim K_1}{\dim H_1} \leq \dfrac{\dim K}{\dim H}$.

The equivalence in stability case is clear by replacing the inequalities by strict inequalities. □