Effective Freeness of Adjoint Linear Systems

Fei YE

Department of Mathematics
The University of Hong Kong

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Conjecture (Fujita 1985)

Let X be a smooth projective variety of dimension n and L be an ample divisor on X. Then $|K_X + mL|$ is base point free if $m \geq n+1$, and $|K_X + mL|$ is very ample if $m \geq n+2$.

Results in lower dimensions

Theorem (Riemann-Roch)

Let C be smooth projective curve and D be a divisor on C. If $\deg D \geq 2g$, then $|D|$ is base point free. If $\deg D \geq 2g+1$, then D is very ample.

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Theorem (Reider 1988)

Let S be a smooth projective surface and L be a line bundle. Then

- $|K_S + L|$ *is base point free if* $L^2 \geq 5$ *and* $L \cdot C \geq 2$ *for any curve* C,
- $|K_S + L|$ *is very ample if* $L^2 \geq 10$ *and* $L \cdot C \geq 3$ *for any curve* C.

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Let X be a smooth projective variety of dimension 4 and L be an ample line bundle on X. Assume that $L^4 \geq 5$, $L^d \cdot Z \geq 5^d$ for any d-dimensional sub-variety Z. Then $|K_X + L|$ is base point free.

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- Fujita 1993: If $L^3 \geq 51$, $L^2 \cdot S \geq 7$ and $L \cdot C \geq 3$, then $|K_X + L|$ is base point free.

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- Fujita 1993: If $L^3 \geq 51$, $L^2 \cdot S \geq 7$ and $L \cdot C \geq 3$, then $|K_X + L|$ is base point free.
- Kawamata 1999: If $L^3 > 27$, $L^2 \cdot S \geq 9$ and $L \cdot C \geq 3$, then $|K_X + L|$ is base point free.

Results in higher dimensions

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Theorem (Kollár 1993)

$|2(n+1)(n+2)!(K_X + (n+2)L)|$ *is base point free.*

Theorem (Angehrn-Siu 1995)

If $m \geq \frac{1}{2}n(n+1)+1$, *then* $|K_X + mL|$ *is base point free.*

Results in higher dimensions

Let X be a smooth projective variety of dimension n and L be an ample line bundle on X.

Theorem (Helmke 1999)

Assume that

1. $L^n > n^n$, *and* $L^{n-1} \cdot H \geq n^{n-1}$ *for all hypersurface* H*;*
2. $L^d \cdot Z \geq \binom{n-1}{d-1} \cdot n^d$ *for all* $Z \subset X$ *of dimension* d*.*

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Theorem (Heier 2002)

If $m \geq (e + \frac{1}{2})n^{\frac{4}{3}} + n^{\frac{2}{3}} + 1$, *where* $e \approx 2.718$ *is the Euler's number, then* $|K_X + mL|$ *is base point free.*

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Conjecture (Stronger Fujita Conjecture)

If $L^n > n^n$ *and* $L^d \cdot Z \geq n^d$ *for any subvariety* $Z \subset X$ *of dimension* d*. Then* $|K_X + L|$ *is base point free.*

Multiplier Ideals and Nadel vanishing theorem

Definition (Multiplier idea sheaf)

Let G be an effective $\mathbb{Q}$-divisor on X. The multiplier ideal of G of weight λ is defined as $\mathcal{I}(\lambda G) = f_*\mathcal{O}_Y(K_{Y/X} - [f^*(\lambda G)])$, where $f : Y \to X$ is a log resolution of D. Denote by $Z(\lambda G)$ the reduced scheme defined by $\mathcal{I}(\lambda G)$.

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Theorem (Nadel vanishing theorem)

Assume that A is an integral divisor such that $A - (K_X + G)$ is nef and big. Then $H^i(X, \mathcal{O}_X(A) \otimes \mathcal{I}(G)) = 0$ for $i > 0$.

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Corollary

Assume that G is an effective $\mathbb{Q}$-divisor linearly equivalent to λL for $\lambda < 1$. If $Z(G)$ is a point, then $|K_X + L|$ is base point free at x.

Two approaches to effective freeness

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The problem in the first approach is that we have to work on singular varieties.

Critical Varieties

Let x be a point in X which is contained in $Z(G)$. We say that G is critical at x if :

1. $x \notin Z((1-\varepsilon)G)$ for any $0 < \varepsilon < 1$;
2. $K_{Y/X} - [f^*G] = P - F - N$ for a log resolution $f : Y \to X$ such that P, F and N are effective with no common component, F is irreducible and $N \cap f^{-1}(x) = \emptyset$.

The component F is called the critical component of G at x and $f(F) = Z$ is called the critical variety of G at x.

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Assume that $G = \lambda L$ is ample. By a tie-breaking technique, we can assume that G is critical at x. Moreover, we can assume that the critical variety Z is the minimal log canonical center of G.

Properties of critical varieties

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- In the case that Z is a surface, $mult_x Z \leq e - 1$, e is the minimal embedding dimension of Z.
- For $d = \dim Z > 2$, we have $mult_x Z \leq \binom{n-1}{d-1}$ (Helmke 1997).

Deficit functions

Let $\pi : Y' \to X$ be the blowing up of x and E be the exceptional divisor. Let $g : Y \to Y'$ be a log resolution of $\pi^*(G) + E$. We then have a log resolution $f = g \circ \pi : Y \to X$. Write $f^*(G) = \sum g_j F_j$, $K_{Y/X} = \sum b_j F_j$, $K_Y - f^*(K_X + G) = \sum a_j F_j$ and $g^*(E) = e_j F_j$.

Definition

If $x \notin Z(G)$, we define the deficit of G at x as

$$def_x(G) := \min_{f(F_j)=x} \{\frac{a_j + 1}{e_j}\};$$

if G is log canonical at x, we define

$$def_x(G) := \lim_{t \to 0^+} def_x((1-t)G).$$

Properties of deficit functions

The deficit of G can also be defined as

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The larger the deficit, the larger the dimension of Z.

Upper bounds of deficit functions

Proposition (Ein, Helmke)

Assume that G is critical at x, Z is the critical variety of G at x and $def_x(G) > 0$. Let e be the minimal embedding dimension of Z at x, $d = \dim Z$ and $m = mult_x Z$. Then

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Let $\alpha(m) = \min\{\alpha \in \mathbb{Z} \cap [0, d] \mid \binom{e-\alpha}{e-d} \geq m\}$.

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Assume that D_1 and D_2 are nice liftings of B. If $x \in Z((1-s)D_1 + (1-t)G)$ for all $0 < s, t \ll 1$, then $x \in Z((1-s')D_2 + (1-t')G)$ for all $0 < s', t' \ll 1$. Moreover, $Z((1-s')D_2 + (1-t')G)$ is a proper closed subset of the critical variety Z of G.

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If $mult_x(B) > def_x(G)$, then for any nice lifting D of B, $x \in Z(D + (1-t)G)$ for all sufficiently small positive number t. Moreover, if $G' = D + (1-t)G$ is log canonical at x, then $def_x(G') \leq def_x((1-t)G) - mult_x(B)$.

Helmke's induction criterion

Let X be a smooth projective variety, L be an ample line bundle over X and G be a $\mathbb{Q}$-divisor linearly equivalent to λL for some positive rational number $\lambda < 1$. Assume that G is critical at x with $def_x(G)$. Let Z be the critical variety of G at x and $d = \dim Z$. If

$$L^d \cdot Z > (\frac{def_x(G)}{1-\lambda})^d \cdot mult_x(Z) \tag{2}$$

then there is a $\mathbb{Q}$-divisor G_1 linearly equivalent to $\lambda' L$ with $\lambda < \lambda' < 1$ such that G' is ciritical at x with the critical variety Z' which is properly contained in Z and

$$\frac{def_x(G')}{1-\lambda'} < \frac{def_x(G)}{1-\lambda}.$$

Sketch of the proof

Step 1: Using the inequality (2), we may choose a divisor B on Z such that $mult_x B > def_x(G)$.

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Step 2: Since G is ample, we can find a divisor D which is a nice lifting of B such that $mult_x D = mult_x B$.

Step 3: Then $p \in Z((1-t)G + D)$ for sufficiently small t. If $(1-t)G + D$ is not critical at p, the replace D by $(1-s)D$ for some s such that $(1-t)G + (1-s)D$ is critical at p. The inequality follows easily.

Step one for induction

Lemma

Let X be a smooth projective variety of dimension n, L be an ample line bundle on X and $x \in X$ be a point. Assume that $\sqrt[n]{L^n} > \sigma_n \geq n$. Then there is an effective divisor G linearly equivalent to $\lambda_0 L$ for $\lambda_0 < 1$ and critical at x such that

$$\frac{\sigma_n \alpha(m)}{\sigma_n - n + \alpha(m)} > \frac{def_x(G)}{1 - \lambda_0}.$$

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Theorem

Let X be a smooth projective variety of dimension 3 and L be an ample line bundle on X. If $\sqrt[3]{L^3} \geq 4$, $\sqrt{L^2 \cdot S} \geq 3$ for any irreducible surface $S \subset X$, and $L \cdot C \geq 3$ for any curve $C \subset X$, then $|K_X + L|$ is base point free.

Volumes of line bundles along sub-varieties

We define the volume of L along the ideal sheaf I as

$$Vol(L, I, \beta) := \lim_{k \to \infty} \frac{n! h^0(X, kL \otimes I^{\lceil \beta k \rceil})}{k^n}.$$

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Let Z be a subvariety and $A_k^{\beta k} = H^0(X, kL \otimes I_Z^{\lceil \beta k \rceil})$.

$M_Z(\beta) := \lim_{k\to\infty} \inf\{\frac{mult_Z s}{k} | s \in A_k^{\beta k}\}$ is concave up.

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$$Vol(\alpha, \beta, L) := Vol(L, I_x, \alpha) - Vol(L, I_x, \beta).$$

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$$Vol(\alpha, \beta, L) := Vol(L, I_x, \alpha) - Vol(L, I_x, \beta).$$

Proposition

Let $\alpha, \beta, \in \mathbb{R}_{\geq 0}$. Assume that $Vol(L, I_x, t) \geq 0$ for $\beta \geq t \geq \alpha$. If Z is a Cartier divisor on X and $m = mult_x(Z)$. Then

$$Vol(\alpha, \beta, L) \leq n \int_\alpha^\beta (t - mM_Z(t))^{n-1} dt.$$

An improvement in codimension one

Let X be a smooth projective variety of dimension n and L be an ample line bundle over X such that $L^n > n^n$. Assume that G be a $\mathbb{Q}$-divisor linearly equivalent to λL for some positive rational number $\lambda < 1$ such that G is critical at x. Let Z be the critical variety of G at x. If Z is a divisor and $L^{n-1} \cdot Z \geq n^{n-1}$, then

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In this case, Z is in the fixed part of the linear system $|\lambda L \otimes I_x^{\lceil \lambda n \rceil}|$ and we can get a smaller upper bounds on $def_x(G)$.

Effective freeness on 4-folds and 5-folds

Theorem

Let X be a smooth projective variety of dimension 4 and L be an ample line bundle on X such that the following conditions holds

1. $\sqrt[d]{L^d \cdot Z} \geq 5$ *for any subvariety* $Z \subseteq X$ *of* $\dim Z = 1, 3, 4$,
2. $\sqrt{L^2 \cdot S} \geq 6$ *for any irreducible surface* $S \subset X$.

Then $|K_X + L|$ *is base point free.*

Theorem

Let X be a smooth projective variety of dimension 5 and L be an ample line bundle on X such that and $\sqrt[d]{L^d \cdot Z} \geq 8$ *for any subvariety* $Z \subseteq X$ *of* $\dim Z = d$, *where* $d = 1, 2, 3, 4, 5$. *Then* $|K_X + L|$ *is base point free at* x.

An example of proofs

Choose a divisor $D \in |L|$ such that $mult_x D > 8^5$. Then $x \in Z(\lambda D)$ for some $\lambda \in (0,1) \cap \mathbb{R}$.

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By tie-breaking, we may assume that $G = \lambda D$ is critical with the critical variety Z. If $\dim Z = d$, then $\frac{8\alpha(m)}{3+\alpha(m)} > \frac{def_x(G)}{1-\lambda_0}$. Apply Helmke's induction to create a new divisor G' with smaller critical variety Z'.

An example of proofs

Choose a divisor $D \in |L|$ such that $mult_x D > 8^5$. Then $x \in Z(\lambda D)$ for some $\lambda \in (0,1) \cap \mathbb{R}$.

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Check if the induction criterion is satisfied to run further induction. For example, if in the previous step, $\dim Z = 4$, $mult_x Z = 1$ and $\dim Z' = 2$, then $\alpha(m) \leq 4$ and $mult_x Z' \leq 3$. Thus $\frac{def_x(G')}{1-\lambda'}\sqrt{3} < \frac{def_x(G)}{1-\lambda}\sqrt{3} < \frac{32\sqrt{3}}{7} < 8$.

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When the critical variety is a point, we can apply Nadel's vanishing theorem to prove the freeness of $|K_X + L|$.

Bounds on pluricanonical maps

Theorem (Hacon-M^{c}Kernan 2006)

Let X be a smooth variety of general type of dimension n. There exist r_n such that $\phi_r : X \dashrightarrow \mathbb{P}H^0(X, \mathcal{O}(rK_X))$ is birational for all $r \geq r_n$.

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The idea to prove this theorem is to find a divisor $D_{x,y}$ linearly equivalent to λK_X for $\lambda < r_n$ such that $x, y \in Z(D_{x,y})$ and x is an isolated point. The way is to apply Angehrn-Siu's idea.

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Effective lower bounds of r have been obtained for 3-folds (Chen-Chen 2008 etc.) and 4-folds (Di Biagio 2010 etc.) using different methods.

Some questions

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Is $|K_X + mL|$ free for $m \geq 4n + 1$?

Thank you!