Decomposition of Higher Jacobian Ideals

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Outline

1. Introduction

1.1 Nash Blowup

- Let X be an algebraic variety an algebraically close field k with $\dim X = n$, and $X_{\mathrm{sm}} \subseteq X$ the smooth locus of X.
- Denote by T_X the tangent sheaf and Ω_X sheaf of Kähler differentials.
- Consider the morphism: $X_{\mathrm{sm}} \xrightarrow{\sigma} \mathbb{P}(\wedge^n \Omega_X), x \mapsto \left(x, \wedge^n \Omega_{X,x}\right)$

Definition 1.1 The Nash blowup of X is defined as $\mathrm{Nash}(X) = \overline{\sigma(X_{\mathrm{sm}})}$, the closure of $\sigma(X_{\mathrm{sm}})$.

- The morphism $\pi|_{\mathrm{Nash}(X)} : \mathrm{Nash}(X) \longrightarrow X$ is birational, where $\pi : \mathbb{P}(\wedge^n \Omega_X) \longrightarrow X$ is the projection.

1.1 Nash Blowup

- If $X \subset \mathbb{A}^N$, then $\tilde{\sigma} : X_{\mathrm{sm}} \longrightarrow X \times \mathrm{Grass}(T_{\mathbb{A}^N}, n)$, $x \mapsto \left(x, T_{X,x}\right)$ is a morphism.

Proposition 1.2 (See for example Ein, De Fernex, Ishii (2008))

$$\mathrm{Nash}(X) = \overline{\tilde{\sigma}(X_{\mathrm{sm}})}$$

Theorem 1.3 (Nobile (1975)) A Nash blowing-up is the blowup of a suitable ideal (the Jacobian ideal if X is a hypersurface).

1.1 Nash Blowup

Conjecture 1.4 Assume $\mathrm{char}(k) = 0$.

1. X is desingularized by the iteration of finitely many Nash blowups.
2. X is desingularized by the iteration of finitely many normalized Nash blowups.

- True for $\dim X = 1$ and part 2. is true for $\dim X = 2$ (see Spivakovsky (1990) and reference therein).
- As far as I know, little is known in dimension 3.
- Both fails if $\dim X \geq 4$ (see Castillo, Duarte, Leyton-Álvarez, Liendo (2024)).

1.2 Nash Blowup of a Coherent Sheaf

Let $\mathcal{M}$ be a coherent sheaf on X.

Definition 1.5 (Oneto, Zatini (1991)) Assume that $\mathcal{M}$ is locally free of rank r over an open dense $U \subset X$. The Nash blowup $\mathrm{Nash}(X, \mathcal{M})$ of X with respect to $\mathcal{M}$ is the Zariski closure $\overline{\sigma(U)}$, where

$$\begin{tikzcd}
 & \mathrm{Grass}(\mathcal{M}, r) \arrow[d, "\pi"] & \overline{\sigma(U)} = \mathrm{Nash}(X, \mathcal{M}) \arrow[l, hook'] \arrow[dl, two heads] \\
U \arrow[ur, "\sigma"] \arrow[r, hook, "i = \pi \circ \sigma"'] & X &
\end{tikzcd}$$

$\pi|_{\mathrm{Nash}(X,\mathcal{M})}$ is birational

1.2 Nash Blowup of a Coherent Sheaf

Proposition 1.6 (Universal Property (Oneto, Zatini (1991))) If $h : Y \longrightarrow X$ is a morphism such that $h^*\mathcal{M}/\ \mathrm{torsion}$ is locally free, then there exists a unique morphism $\nu : Y \longrightarrow \mathrm{Nash}(X, \mathcal{M})$ such that

$$\nu = \pi\big|_{\mathrm{Nash}(X,\mathcal{M})} \circ h.$$

- Let $\varphi : \wedge^r \mathcal{M} \to \wedge^r \mathcal{M} \otimes \mathcal{R}(X) \overset{\sim}{\to} \mathcal{R}(X)$ and $\mathcal{E} = \varphi(\wedge^r \mathcal{M})$, where $\mathcal{R}(X)$ is the sheaf of total quotient rings of X.

Theorem 1.7 (Oneto, Zatini (1991))

$$\mathrm{Bl}_{\mathcal{E}}(X) \overset{\sim}{\to} \mathrm{Nash}(X, \mathcal{M}).$$

1.3 Higher Nash Blowups

- Let $x \in X$ be a point and $x^{(k)}$ the n-th infinitesimal neighborhood. If x is smooth, then

$$\sigma_k : X_{\mathrm{sm}} \longrightarrow \mathrm{Hilb}_{\binom{n+k}{k}}(X)$$
$$x \mapsto \left[x^{(n)}\right]$$

is a morphism to the Hilbert scheme of $\binom{n+k}{k}$ point.

Definition 1.8 (Yasuda (2007)) The k-th Nash blowup $\mathrm{Nash}_k(X)$ of X is the closure $\overline{\Gamma_k(X_{\mathrm{sm}})}$, where

$$\Gamma_k = \mathrm{id} \times \sigma_k : X_{\mathrm{sm}} \longrightarrow X \times \mathrm{Hilb}_{\binom{n+k}{k}}(X).$$

1.3 Higher Nash Blowups

Proposition 1.9 (Yasuda (2007)) If $\mathrm{char}(k) = 0$, then,

$$\mathrm{Nash}_k(X) \xrightarrow{\sim} \overline{\sigma_k(X_{\mathrm{sm}})}.$$

- Let $\Delta \hookrightarrow X \times X$ be the diagonal and $\mathcal{J}$ the ideal sheaf defining Δ. Denote by $\mathcal{P}^k_X = \mathcal{O}_{X\times X}/\mathcal{J}^{k+1}$ the sheaf of principle part of order k and $\Omega^{(k)}_X = \mathcal{J}/\mathcal{J}^{k+1}$ the sheaf of differentials of order k.

Proposition 1.10 (Yasuda (2007); Lê, Yasuda (2024))

$$\begin{aligned}\mathrm{Nash}(X,k) &= \mathrm{Nash}(X,\mathcal{P}^k) = \mathrm{Nash}\left(X, \wedge^{\binom{n+k}{k}} \mathcal{P}^{(k)}_X\right) \\ &= \mathrm{Nash}\left(X, \Omega^{(k)}_X\right) = \mathrm{Nash}\left(X, \wedge^{\binom{n+k}{k}-1} \Omega^{(k)}_X\right).\end{aligned}$$

1.4 Fitting Ideals

Definition 1.11 (Lê, Yasuda (2024); Nguyen (2024)) The k-th Jacobian ideal is the smallest Fitting ideal

$$\mathcal{J}_k = \mathbf{Fitt}_{\binom{n+k}{k}-1}\Omega_X^{(k)}.$$

Proposition 1.12 (Lê, Yasuda (2024); Nguyen (2024)) If X is a complete intersection, then

$$\mathrm{Nash}(X, k) = \mathrm{Bl}_{\mathcal{J}_k} X.$$

- Explicit presentations of $\Omega_X^{(k)}$ were obtained: Duarte (2017) for hypersurface, Barajas, Duarte (2020) for complete intersection.

1.5 Higher Nash Blowup Algebras

- Let $F \in \mathcal{O}_n$ be an analytic germ with $F(0) = 0$.

Definition 1.13 (Hussain, Ma, Yau, Zuo (2023)) The k-th Nash blowup algebra is $\mathcal{T}_k(F) = \mathcal{O}_n/(F + \mathcal{J}_k(F))$.

Theorem 1.14 (Mather, Yau (1982)) For $F, G \in \mathcal{O}_n$, $\mathcal{T}_1(F) = \mathcal{T}_1(G)$ iff $(F, 0)$ and $(G, 0)$ are biholomorphic.

Remark 1 The equivalence fails over $\mathbb{R}$ and $\mathrm{char}(k) > 0$. (Mather, Yau (1981)), but under additional conditions (Greuel, Pham (2016)).

1.5 Higher Nash Blowup Algebras

Remark 2

1. $\mathcal{T}_k(F)$ is also called the moduli algebra (Mather, Yau (1981)).
2. $\mathcal{T}_1(F)$ coincides with the Tjurina algebra
3. Let $\mathcal{M}_k(F) := \mathcal{O}_n/(\mathcal{J}_k(F))$. The algebra $\mathcal{M}_1(F)$ coincides with the Milnor algebra. But $\mathcal{M}_k(F)$ is different from the k-th Milnor algebra defined in Dimca, Gondim, Ilardi (2020).
4. Mather, Yau (1981) also showed that $\mathcal{M}_k(F) = \mathcal{M}_k(G)$ iff F and G are biholomorphic.

1.5 Higher Nash Blowup Algebras

Conjecture 1.15 (Hussain, Ma, Yau, Zuo (2023)) $\mathcal{O}_n/(F+\mathcal{J}_k(F))$ are contact invariants.

- For $n=2, k=2$: Hussain, Ma, Yau, Zuo (2023)
- For $n=3, k=2$: Shen, Ramirez, Ye (2025)
- For $k=2$: Ye (2025)
- Lê, Yasuda (2024, Theorem 2.5): Conjecture 1.15 holds true.
- Nguyen (2024, Theorem 3.2): The converse of Conjecture 1.15 also holds true.

1.6 Higher Jacobian Matrices

Definition 1.16 (k-th Jacobian Matrix[1]) The k-th Jacobian matrix of a function $F \in \mathcal{O}_n$ is

$$\mathrm{Jac}_k(F) = \left(r_{\beta,\alpha}\right)_{0 \leq |\beta| \leq k-1,\, 1 \leq |\alpha| \leq k},$$

where $\alpha \in \mathbb{N}^{|\alpha|}$, $\beta \in \mathbb{N}^{|\beta|}$, and

$$r_{\beta,\alpha} := \begin{cases} \frac{\partial^{\alpha-\beta} F}{(\alpha-\beta)!} & \text{if } \alpha > \beta \\ 0 & \text{otherwise,} \end{cases}$$

where $\partial^{\alpha} = \partial^{\alpha_1} \cdots \partial^{\alpha_d}$ and $\alpha! = \alpha_1! \cdots \alpha_d!$ for $\alpha = (\alpha_1, ..., \alpha_d)$.

[1] This definition differs from Duarte (2017) when $|\beta| = 0$ and $|\alpha| = 1$.

1.6 Higher Jacobian Matrices

Example 1 Let $F = x^2 + y^2$. Then

$$\mathrm{Jac}_2(F) = \begin{pmatrix} 2x & 2y & 1 & 0 & 1 \\ 0 & 0 & 2x & 2y & 0 \\ 0 & 0 & 0 & 2x & 2y \end{pmatrix}$$

$$\mathrm{Jac}_3(F) = \begin{pmatrix} 2x & 2y & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2x & 2y & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2x & 2y & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2x & 2y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2x & 2y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2x & 2y \end{pmatrix}$$

Proposition 1.17 (Lê, Yasuda (2024)) The k-th Jacobian ideal $\mathcal{J}_{k(F)}$ of F is the ideal of maximal minors of $\mathrm{Jac}_k(F)$.

- Let $Q(F) = \partial_{xx}F\left(\partial_y F\right)^2 - 2\partial_{xy}F\partial_x F\partial_y F + \partial_{yy}F(\partial_x F)^2$

Example 2 Let $F = x^2 + y^2$. Then

$$\mathcal{J}_2(F) = \left(x^3, xy^2, x^2y, y^3, x^2 + y^2\right) = \mathcal{J}_1(F)^3 + (Q(F))$$

Example 3 Let $F = x^3 - y^2$. Then

$$\mathcal{J}_2(F) = \left(x^6, x^2y^2, x^4y, y^3, 4xy^2 - 3x^4\right) = \mathcal{J}_1^3 + (Q(F))$$

Example 4 Let $F = x^2 + y^2$. Then

$$\mathcal{J}_3(F) = \mathcal{J}_1(F)^6 + \mathcal{J}_1^3(F)(Q(F)) + (xy, x^2 - y^2)(Q(F))$$

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2. Structure of $\mathcal{J}_2(F)$

2.1 Main Results

Conjecture 2.1 (Ramirez, Shen, Ye (2024))

$$\mathscr{J}_2(F) = \mathscr{J}_1(F)^{n+1} + \mathscr{J}_1(F)^{n-2}\mathscr{Q}(F),$$

where $\mathscr{Q}(F) = \big(Q_{ij;kl}(F)\big)$ and

$$\begin{aligned} Q_{ij;kl}(F) = & \ \partial_{ik}F\partial_j F\partial_l F - \partial_{il}F\partial_j F\partial_k F \\ & -\partial_{jk}F\partial_i F\partial_l F + \partial_{jl}F\partial_i F\partial_k F. \end{aligned}$$

Main Theorem (Ye (2025)) Conjecture 2.1 is true.

- Hussain, Ma, Yau, Zuo (2023) for $n = 2$, Ramirez, Shen, Ye (2024) for $n = 3$

2.1 Main Results

Proposition 2.2 Conjecture 1.15 is true for $k = 2$.

Corollary 2.2.1

$$\mathscr{J}_2(F) \subset \mathscr{J}_1(F)^n.$$

Remark 3 It was proved by Lê, Yasuda (2024) that

$$\mathscr{J}_k(F) \subset \mathscr{J}_1(F)^{\binom{n+k-2}{n-1}}.$$

2.2 Examples and Lemmas

- Write $f_i := \partial_i F$, $f_{ij} := \partial_{ij} F$, and $Q_{ij;kl} := Q_{ij;kl}(F)$.
- Let $\beta_i = \left(0, \ldots, 0, \overset{i\text{-th}}{1}, 0, \ldots, 0\right)$ and $\beta_{i,j} = \beta_i + \beta_j$.
- Rows and columns of $\mathrm{Jac}_2(F)$ can be labeled using β_i and β_{ij} respectively.
- With the above notations, the (β_k, β_{ij}) entry of $\mathrm{Jac}_2(F)$ is

$$\mathrm{Jac}_2(F)(\beta_k, \beta_{ij}) = \begin{cases} f_j & \text{if } i = k \\ f_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

2.2 Examples and Lemmas

Example 5 (Powers of monomials are determinant of submatrices)

$$
\begin{array}{c}
 & \begin{array}{cccccccc} \beta_{11} & \beta_{12} & \beta_{23} & \beta_{45} & \beta_{55} & \beta_{56} & \beta_{67} & \beta_{78} \end{array} \\
\begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \end{array} &
\begin{pmatrix}
f_1 & f_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & f_1 & f_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & f_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & f_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & f_4 & f_5 & f_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & f_5 & f_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & f_6 & f_8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & f_7
\end{pmatrix}
\end{array}
$$

2.2 Examples and Lemmas

Lemma 2.3 Let $M = \mathrm{Jac}_2(F)\Big([\beta_{i_1}, ..., \beta_{i_r}], [\beta_{i_1,j_1}, ..., \beta_{i_r,j_r}]\Big)$ be as submatrix of the second Jacobian matrix $\mathrm{Jac}_2(F)$. If

$$i_1 < ..., < i_r \quad \text{and} \quad j_1 \leq ... \leq j_r,$$

Then

$$\det M = \prod_{i=1}^{r} f_{j_r}.$$

2.2 Examples and Lemmas

Example 6 (Powers of monomials are determinant of submatrices)

$$
\begin{array}{c|ccccccccc}
 & \beta_{11} & \beta_{14} & \beta_{46} & \beta_{66} & \beta_{77} & \beta_{13} & \beta_{12} & \beta_{15} & \beta_{38} \\
\hline
\beta_0 & \frac{1}{2}f_{11} & f_{14} & f_{46} & \frac{1}{2}f_{66} & \frac{1}{2}f_{77} & f_{13} & f_{12} & f_{15} & f_{38} \\
\beta_1 & f_1 & f_4 & 0 & 0 & 0 & f_3 & f_2 & f_5 & 0 \\
\beta_4 & 0 & f_1 & f_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta_6 & 0 & 0 & f_4 & f_6 & 0 & 0 & 0 & 0 & 0 \\
\beta_7 & 0 & 0 & 0 & 0 & f_7 & 0 & 0 & 0 & 0 \\
\beta_3 & 0 & 0 & 0 & 0 & 0 & f_1 & 0 & 0 & f_8 \\
\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & f_1 & 0 & 0 \\
\beta_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_1 & 0 \\
\beta_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_3
\end{array}
$$

2.2 Examples and Lemmas

Lemma 2.4 Let M be a maximal square submatrix of $\mathrm{Jac}_2(F)$, then M is permutation equivalent to a block upper triangular matrix, i.e.,

$$M = \begin{pmatrix} D & * & * & * \\ 0 & B_m & * & * \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & B_1 \end{pmatrix}.$$

2.3 Minors of $\mathrm{Jac}_2(F)$

Example 7 Case: $(i-j)(i-j)$

$$\det\begin{pmatrix} \frac{1}{2}f_{ii} & f_{ij} & \frac{1}{2}f_{jj} \\ f_i & f_j & 0 \\ 0 & f_i & f_j \end{pmatrix} = \frac{1}{2}Q_{ij;ij}$$

Case: $(i-j)(k-l)$

$$\det\begin{pmatrix} f_{ik} & f_{jk} & f_{il} & f_{jl} & f_{lm} \\ f_k & 0 & f_l & 0 & 0 \\ 0 & f_k & 0 & f_l & 0 \\ f_i & f_j & 0 & 0 & 0 \\ 0 & 0 & f_i & f_j & f_m \end{pmatrix} = f_m f_l Q_{ij;kl}$$

2.3 Minors of $\mathrm{Jac}_2(F)$

Example 8 Case: $(i-j)(k-l)-il$

$$\det\begin{pmatrix} f_{ik} & f_{jk} & f_{jl} & f_{im} & \frac{1}{2}f_{ll} & f_{lm} \\ f_k & 0 & 0 & f_m & 0 & 0 \\ 0 & f_k & f_l & 0 & 0 & 0 \\ f_i & f_j & 0 & 0 & 0 & 0 \\ 0 & 0 & f_j & 0 & f_l & f_m \\ 0 & 0 & 0 & f_i & 0 & f_l \end{pmatrix} - \det\begin{pmatrix} f_{jj} & f_{il} & f_{kk} & f_{im} & \frac{1}{2}f_{ll} & f_{lm} \\ 0 & f_l & 0 & f_m & 0 & 0 \\ f_j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_k & 0 & 0 & 0 \\ 0 & f_i & 0 & 0 & f_l & f_m \\ 0 & 0 & 0 & f_i & 0 & f_l \end{pmatrix}$$

$$= \det\begin{pmatrix} f_m & 0 & 0 \\ 0 & f_l & f_m \\ f_i & 0 & f_l \end{pmatrix} Q_{ij;kl}(F) = f_m f_l^2 Q_{ij;kl}(F)$$

2.3 Minors of $\mathrm{Jac}_2(F)$

Example 9 Case: $(i-j)(i-k)-ik$

$$\det\begin{pmatrix} \frac{1}{2}f_{ii} & f_{ij} & f_{jk} & f_{kl} & f_{jl} \\ f_i & f_j & 0 & 0 & 0 \\ 0 & f_i & f_k & 0 & f_l \\ 0 & 0 & f_j & f_l & 0 \\ 0 & 0 & 0 & f_k & f_j \end{pmatrix} + \det\begin{pmatrix} \frac{1}{2}f_{ii} & f_{ik} & \frac{1}{2}f_{jj} & f_{kl} & f_{jl} \\ f_i & f_k & 0 & 0 & 0 \\ 0 & 0 & f_j & 0 & f_l \\ 0 & f_i & 0 & f_l & 0 \\ 0 & 0 & 0 & f_k & f_j \end{pmatrix}$$

$$= \det\begin{pmatrix} f_l & 0 \\ 0 & f_l \end{pmatrix} Q_{ij;ik}(F) - \det\begin{pmatrix} 0 & f_l \\ f_k & f_j \end{pmatrix} Q_{ij;ij}(F)$$

$$= f_j f_l Q_{ij;ik}(F) + f_k f_l Q_{ij;ij}(F)$$

2.3 Minors of $\mathrm{Jac}_2(F)$

Lemma 2.5 Given a maximal square submatrix M of $\mathrm{Jac}_2(F)$. One of the following holds true:

1. $\det(M) = 0$,
2. $\det(M)$ is in the form $Q \cdot P \cdot \det(A)$, where, $Q \in \mathcal{Q}(F)$, P is a monomial in $\mathcal{J}_1(F)^p$, and A is a submatrix such that $\det(A) \in \mathcal{J}_1(F)^{n-2-p}$.
3. There exists another matrix N such that $\det(M) + \det(N)$ is in the form $Q_A \cdot P_A \cdot \det(A) + Q_B \cdot P_B \cdot \det(B)$, where $Q_A, Q_B \in \mathcal{Q}(F)$, $P_A \in \mathcal{J}_1(F)^p$, $P_B \in \mathcal{J}_1(F)^q$, and A and B are submatrices such that $\det(A) \in \mathcal{J}_1(F)^{n-2-p}$ and $\det(B) \in \mathcal{J}_1(F)^{n-2-q}$.

3. Ideas of the Proofs

3.1 Proof of the Main Theorem

- Let M be a maximal submatrix of $\mathrm{Jac}_2(F)$.

Lemma 2.3 implies the following proposition.

Proposition 3.1

1. If M contains a column β_{0i}, then $\det(M) \in \mathcal{J}_1(F)^{n+1}$.
2. $\mathcal{J}_1(F)^{n+1} \subset \mathcal{J}_2(F)$.

Lemma 2.5 implies the following proposition.

Proposition 3.2

1. If M contains no column β_{0i}, then $\det(M) \in \mathcal{Q}(F)\mathcal{J}_1(F)^{n-2}$.
2. $\mathcal{Q}(F)\mathcal{J}_1(F)^{n-2} \subset \mathcal{J}_2(F)$.

3.2 Vectorization and Tensor

Let E_{ij} be the skew-symmetric matrix whose only nonzero entries are the (i,j) and (j,i) entries, moreover, the (i,j) entry is 1 if $i<j$.

Lemma 3.3

$$Q_{ij;kl}(F) =$$

$$\mathrm{Vec}\left(\mathrm{Hess}(F)^T\right)\cdot E_{ij}\otimes E_{kl}\cdot \mathrm{Vec}\Big(\mathrm{Jac}(F)^T\ \mathrm{Jac}(F)\Big),$$

where $\mathrm{Jac}\,(F)=\nabla(F)^T$ is the Jacobian matrix, $\mathrm{Hess}(F)=\mathrm{Jac}^T(\mathrm{Jac}(F))=\mathrm{Jac}(\nabla(F))^T$ is the Hessian matrix, and $\mathrm{Vec}(M)$ is the vectorization of the matrix M which is a column vector obtained by stacking column vectors of M.

3.2 Vectorization and Tensor

Example 10 For $F = x^2 + y^2 \in \mathbb{C}\{x, y\}$,

$$\mathrm{Jac}_1(F) = (2x, 2y), \quad \mathrm{Hess}(F) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \text{and}$$

$$\begin{aligned} Q_{12;12} &= 8x^4 + 8y^4 \\ &= \mathrm{Vec}\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right)^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathrm{Vec}\left(\begin{pmatrix} 2x \\ 2y \end{pmatrix} \begin{pmatrix} 2x & 2y \end{pmatrix}\right) \\ &= \begin{pmatrix} 2 & 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 4x^2 \\ 4xy \\ 4xy \\ 4y^2 \end{pmatrix}. \end{aligned}$$

3.3 Chain Rules

Lemma 3.4 (Chain Rule for Jacobian)

$$\mathrm{Jac}(F \circ \varphi) = \mathrm{Jac}(F) \circ \varphi \cdot \mathrm{Jac}(\varphi).$$

Lemma 3.5 (Chain Rule for Hessian) Set $\varphi = (\varphi_1, \ldots, \varphi_n)^T$. Then

$$\begin{aligned} &\mathrm{Hess}(F \circ \varphi) \\ &= \mathrm{Jac}(\varphi)^T \Big(\mathrm{Hess}(F) \circ \varphi\Big) \mathrm{Jac}(\varphi) + \sum_{k=1}^{n} \Big(\partial_k F \circ \varphi \ \mathrm{Hess}(\varphi_k)\Big). \end{aligned}$$

3.4 HMYZ Conjecture

- The product rule implies

Lemma 3.6 If $u \in \mathcal{O}_n$ is a unit, then

$$\mathcal{J}_2(uF) = \mathcal{J}_2(F).$$

- Chain rules and the characterization of $Q_{ij;kl}$ leads to

Lemma 3.7

$$\mathcal{J}_1(F \circ \varphi) \subseteq \mathcal{J}_1(F) \circ \varphi.$$

$$\mathcal{Q}(F \circ \varphi) \subseteq \Big(\mathcal{Q}(F) \circ \varphi + \mathcal{J}_1(F)^3\Big) \circ \varphi.$$

- HMYZ conjecture for $k = 2$ follows by applying inverses.

3.5 Questions

Question 1 Does $\mathscr{J}_k(F)$ have a similar decomposition structure?

Question 2 Can one give a proof of the decomposition using properties of Fitting ideals?

Thank you
for your attention!

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