Topics on Complex Manifolds

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Abstract

We will study Hirzebruch's constructions of ball quotient surfaces and related topics which include Miyaoka-Yau inequality, branched coverings, Chern numbers, intersection theory on algebraic surfaces, canonical resolutions of surface singularities, etc. Basic knowledge on complex geometry, algebraic topology, algebraic geometry, commutative algebra and field theory will be very helpful.

1 Metrics, connections and curvatures

In this and the next sections, we closely follow [Kob87].

1.1 Metrics

We first briefly review Hermitian metrics on complex manifolds.

Definition 1.1. For a complex vector bundle E over M, a Hermitian metric h assigns a Hermitian inner products $h_p(\cdot,\cdot)$ on each fiber E_p, i.e. $h_p(\cdot,\cdot)$ is linear in the first parameter, $h_p(w,v) = \overline{h_p(v,w)}$ and $h_p(w,w) > 0$, such that h_p varies smoothly on the point $p \in M$.

Given a local frame $\underline{e} = (e_1,\ldots,e_r)$ over an open subset U, we denote by $h_{i\bar{j}} = h(e_i,e_j)$ which are smooth complex-valued functions. Then we can write $h(w,v) = \underline{w}^T H \underline{\bar{v}}$, where $H = (h_{i\bar{j}})$ is a positive-definite Hermitian matrix, i.e. $\bar{H} = H^T$, and $\underline{e}\cdot\underline{w} = w$. Equivalently, $h(w,\bar{v}) = \underline{w}^T H \underline{v}$. Therefore, h can be viewed as a global smooth complex-valued section of $E^* \otimes \bar{E}^*$. Indeed, with respect to the natural pairing $E \times E^* \to \mathbb{C}$, a Hermitian metric h on a complex vector bundle E on M can be interpreted as a tensor $h = \sum_{i,j=1}^n h_{i\bar{j}} e^i \otimes \bar{e}^j$, where $(e^1,\ldots,e^r)$ is a local frame of E^*, and $(h_{i\bar{j}})$ is an $n\times n$ positive definite Hermitian matrix of complex valued smooth functions.

Definition 1.2. A complex manifold M is called a Hermitian manifold if the complex tangent bundle T_M admits a Hermitian metric g.

Definition 1.3. Let (M,g) be a Hermitian manifold. Denote by $\omega_g = -\frac{1}{2}Im(g) = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^n g_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$. We call ω_g the Kähler form (or fundamental form) of the Hermitian metric g.

The equality makes sense because of the fact that $a \wedge b$ is equivalent to $\frac{1}{2}(a \otimes b - b \otimes a)$.

Example 1.4. There is a flat metric on $\mathbb{C}^n$ whose Kähler form is $\omega = \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$.

Example 1.5 (Fubini-Study metric). On projective space $\mathbb{P}^n$, there is a metric $g = d_{FS}$ given by

$$\omega_g = \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log(1 + \|z\|^2),$$

where $\|z\| = \sum_{i=1}^{n} |z_i|^2$. The metric d_{FS} is called the Fubini-Study metric.

Example 1.6 (Bergman Metric). On the unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$, there is a metric $g = ds^2_{\mathbb{B}^n}$ given by

$$\omega_g = \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log(1 - \|z\|^2).$$

The metric $ds^2_{\mathbb{B}^n}$ is called the Bergman metric.

1.2 Connections and curvatures

Roughly speaking, a connection is a derivative on sections of bundles and the curvature of a connection measure the failure of commutativity of partial derivatives. You may see local and global definitions of connections and curvatures. They are essentially the same, since connections and curvatures behave locally but should extend globally in category of smooth objects (see Remark 1.8).

Here, we use the global definition. We use the following notations.

- $\mathcal{A}^p$ = the sheaf of smooth complex p-forms over M, i.e. $\mathcal{A}^p(U) = \Gamma(U, \wedge^p T_U^*)$.
- $\mathcal{A}^p(E)$ = the sheaf of smooth complex p−forms over M with value in E, i.e. $\mathcal{A}^p(E)(U) = \Gamma(U, \wedge^p T_U^* \otimes E)$.

In other words, $\mathcal{A}^p$ is the sheaf of complex smooth p-forms.

Definition 1.7. Let E be a complex vector bundle on a complex manifold M. A connection ∇ is a $\mathbb{C}$-linear homomorphism $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$

such that

$$\nabla(f\sigma) = df\sigma + f\nabla\sigma,$$

where $f \in \mathcal{A}^0$ and $\sigma \in \mathcal{A}^0(E)$.

Remark 1.8. It is clear that global sections of a vector bundle E are all elements in the sheaf of local sections $\mathcal{A}^0(E)$. Therefore, ∇ also defines over $\Gamma(M, E)$ which is seen in many textbooks. Conversely, since taking derivative is a local manner, given a local section σ_U, we can consider smaller open subsets V so that $\bar{V} \subset U$. By partition of unity, we know that a smooth section on $\bar{V}$ extends to a global section. Therefore, given a connection defined over $\Gamma(M, E)$, we can define a connection on the sheaf $\mathcal{A}^0(E)$ in the following way, $\nabla(\sigma_U) := \nabla(\tilde{\sigma})$, where $\tilde{\sigma}$ is a global section locally extending σ_U (i.e. $\tilde{\sigma} = \sigma_V$ for a open subset $V \subset \bar{V} \subset U$.) See [Lee09] for more discussions.

Let $(e_1, \ldots, e_r)$ be a local frame of E over an open set $U \subset M$. Given a connection ∇, we can write $\nabla e_i = \sum \omega_i^{\ j} e_j$, where $\omega_i^{\ j} = \sum \Gamma_i^{\ j}{}_k \mathrm{d}x^k$ are smooth complex 1-forms. We call the matrix $\omega = \left(\omega_i^{\ j}\right)$ the connection form of ∇. Note that a connection can be viewed as a global smooth section of the sheaf $\mathcal{A}^1(\mathcal{E}nd(E))$.

Exercise 1.9. Let E be a vector bundle of rank r on M and σ is a local frame field such that the connection matrix is ω. Given another local frame field σ' and connection matrix ω' such that $\sigma = \sigma' A$, where A is the transition function, we see that $\omega = A^{-1}\omega' A + A^{-1}\mathrm{d}A$.

Exercise 1.10. Given a connection ∇ on a vector bundle E on M and a section $\xi \in \mathcal{A}^0(E)$, we can we can evaluate the $\nabla\xi$ along a vector field X in T_M in the sense that $\frac{\partial}{\partial x_i}(\mathrm{d}x_j) = \delta_{ij}$. Then $\nabla_X \xi$ is in $\mathcal{A}^0(E)$. Show that $\nabla_X(f\xi) = f\nabla_X\xi + X(f)\xi$.

For $p \geq 1$, we can extend ∇ to a linear homomorphism $\nabla_p : \mathcal{A}^p(E) \to \mathcal{A}^{p+1}(E)$ such that $\nabla_p(f\sigma) = df \otimes \sigma + (-1)^p f \wedge \nabla_p(\sigma)$.

Unlike partial differentiations of functions, in general, partial differentiations of sections of a vector bundle in deferent directions do not commute. The curvature of a connection measures the failure of commutativity.

Definition 1.11. Let E be a complex vector bundle on M with a connection ∇. The curvature Ω of ∇ is $\Omega = \nabla \circ \nabla$. With respect to local frames, the curvature form is given by $\Omega = d\omega - \omega \wedge \omega$, where ω is the connection form of ∇.

Locally, we see that $\Omega = \left(\Omega_i^{\ j}\right)$, where $\Omega_i^{\ j} = \frac{1}{2}\sum R_i^{\ j}{}_{lk} \mathrm{d}x^k \wedge \mathrm{d}x^l$ are 2-forms. A curvature form of a complex manifold M is also called a curvature tensor.

If Ω' is the curvature form of ∇ over another local frame $(e_1', \ldots, e_r')$ and transition matrix is a, then $\Omega = a\Omega' a^{-1}$.

Exterior differentiation of Ω gives the Bianchi identity:

$$\mathrm{d}\Omega = \omega \wedge \Omega - \Omega \wedge \omega.$$

Let X and Y be p-form and q-form respectively. We denote by $[X, Y] = X \wedge Y - (-1)^{pq} Y \wedge X$. Then $\mathrm{d}\Omega = [\omega, \Omega]$.

For manifolds and complex vector bundles with additional properties, we may have connections with special properties.

Definition 1.12. Let E be a complex vector bundle with a Hermitian metric h on a complex manifold M. A connection ∇ on E is compatible with h, if for any smooth sections ξ_1 and ξ_2 of E, we have $\mathrm{d}h(\xi_1, \xi_2) = h(\nabla\xi_1, \xi_2) + h(\xi_1, \nabla\xi_2)$.

Let (E, h) be a Hermitian vector bundle with a connection ∇ compatible with h. Then in matrix notation, we have

$$\mathrm{d}H = \omega H + H\bar{\omega}^T,$$

where M^T means the transpose of the matrix M. Furthermore, we obtain

$$0 = \mathrm{d}(\mathrm{d}H) = \Omega H + H\bar{\Omega}^T.$$

Exercise 1.13. Let (E, h) be a Hermitian vector bundle with a connection ∇ compatible with h. Show that

$$\mathrm{d}H = \omega H + H\bar{\omega}^T$$

and

$$\Omega H + H\bar{\Omega}^T = 0.$$

Now we consider holomorphic vector bundles on a complex manifold M and the holomorphic tangent bundle T_M. Set

- $\mathcal{A}^{p,q}$ = the sheaf of smooth complex (p,q)-forms over M.
- $\mathcal{A}^{p,q}(E)$ = the sheaf of smooth complex $(p,q)-$forms over M with value in E.

Then $\mathcal{A}^k = \oplus_{p+q=k}\mathcal{A}^{p,q}$ and $\mathcal{A}^k(E) = \oplus_{p+q=k}\mathcal{A}^{p,q}(E)$. We can write a connection ∇ on E as $\nabla = \nabla' + \nabla''$, where $\nabla' : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p+1,q}(E)$ and $\nabla'' : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$.

Definition 1.14. A connection ∇ on a holomorphic vector bundle E is compatible with the holomorphic structure if $\nabla'' = \bar{\partial}$.

Proposition 1.15. *Let E be a holomorphic vector bundle with a Hermitian metric h on M. Then there exists a unique connection ∇_h compatible with both the holomorphic structure and the metric h.*

Proof. We first show the uniqueness. Assume that ∇ is a connection on E that is compatible with both the holomorphic structure and h. Let $(e_1, \ldots, e_r)$ be a local holomorphic frame of E. Since $\nabla'' = \bar{\partial}$, then for each holomorphic section s, ∇s is of $(1,0)$-form with value in E. Therefore, each ω_i^j is $(1,0)$-form. By compatibility of ∇ with the Hermitian metric h, we have

$$\mathrm{d}H = \omega H + H\bar{\omega}^T.$$

We take the $(1,0)$-part on both sides. In matrix form, we get $\partial H = \omega H$. Therefore $\omega = \partial H \cdot H^{-1}$. This prove the uniqueness.

For the existence, we define ∇_h by the matrix $\omega = \partial H \cdot H^{-1}$. One can show that this is the unique connection compatible with both the holomorphic structure and the metric h. Q.E.D

Definition 1.16. The unique connection ∇_h compatible with holomorphic structure and the metric h is called a Hermitian connection.

Since ω is a $(1,0)$ form, then the curvature form $\Omega_h = \mathrm{d}\omega - \omega \wedge \omega$ has no $(0,2)$-part. Note that $\Omega = -H\bar{\Omega}^T H^{-1}$. We see that Ω_h also has no $(2,0)$-part. Therefore, Ω_h is of $(1,1)$-form and we obtain that

$$\Omega_h = \mathrm{d}\omega - \omega \wedge \omega = \bar{\partial}(\partial H \cdot H^{-1}).$$

Locally, we can write $\Omega_h = \left(\Omega_i^j\right)$ where $\Omega_i^j = \sum R_{i\ k\bar{l}}^{\ j} dz^k \wedge d\bar{z}^l$. We set $R_{i\bar{j}k\bar{l}} = \sum h_{\alpha\bar{j}} R_{i\ k\bar{l}}^{\ \alpha}$. Denote by $\left(h^{i\bar{j}}\right)$ the inverse matrix of $\left(h_{i\bar{j}}\right)$, i.e $\sum h_{i\bar{j}} h^{k\bar{j}} = \delta_i^k$ the Kronecker delta.

One can also view a curvature as a 2-tensor $\sum \Omega_i^j e^i \otimes e_j$ with values in $\mathcal{E}nd(E)$, where $(e^1, \dots, e^r)$ is the dual of $(e_1, \dots, e_r)$. By identifying E^* with $\bar{E}$ using the Hermitian metric h, the curvature form of a Hermitian vector bundle E, h can be viewed a tensor

$$R = \frac{1}{2} \sum R_{i\bar{j}k\bar{l}} e^i \otimes \bar{e}^j \otimes dz^k \otimes d\bar{z}^l.$$

In particular, the curvature form of the a Hermitian manifold is a $(4,0)$-type tensor

$$R = \frac{1}{2} \sum R_{i\bar{j}k\bar{l}} dz^i \otimes d\bar{z}^j \otimes dz^k \otimes d\bar{z}^l.$$

Note that one can also recognize the curvature tensor R as a map $E \otimes \bar{E} \otimes T_M \otimes \bar{T}_M \to \mathbb{C}$ using the natural duality paring, i.e. $R(e_i, \bar{e}_j, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}) = R_{ijk\bar{l}}$.

Definition 1.17. Let (E, h) be a Hermitian vector bundle on a Hermitian manifold (M, g). We define the Ricci curvature form of h with repeat to g as

$$\mathrm{Ric}_h = \frac{\sqrt{-1}}{2} \sum R_{k\bar{l}} dz^k \wedge d\bar{z}^l,$$

where $R_{k\bar{l}} = \sum g^{i\bar{j}} R_{i\bar{j}k\bar{l}}$.

Note that the above definition generalizes the Ricci curvature form of a Riemannian manifold.

Definition 1.18. Let (M, g) be a Riemannian manifold M. We define the Ricci curvature form of the Levi-Civita connection ∇ as

$$\mathrm{Ric}_g = \frac{\sqrt{-1}}{2} \mathrm{tr}(\Omega) = \frac{\sqrt{-1}}{2} \sum R_{k\bar{l}} dz^k \wedge d\bar{z}^l,$$

where $R_{k\bar{l}} = \sum R_{i\ k\bar{l}}^{\ \bar{i}}$.

Definition 1.19. Let (E, h) be a Hermitian vector bundle on a Hermitian manifold (M, g). We define the mean curvature form of h with repeat to g as

$$\hat{K}_h = \frac{\sqrt{-1}}{2} \sum K_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where $K_{i\bar{j}} = \sum g^{k\bar{l}} R_{i\bar{j}k\bar{l}}$.

Note that on Kähler manifolds, we have $\hat{K}_h = \mathrm{Ric}_h$.

Exercise 1.20. Show that the Riemann-Christoffel symbol $R_{i\bar{j}k\bar{l}}$ of the curvature of the Hermitian connection is given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum h^{\alpha\bar{\beta}} \frac{\partial h_{i\bar{\beta}}}{\partial z_k} \frac{\partial h_{\alpha\bar{j}}}{\partial \bar{z}^l}.$$

Exercise 1.21. Let $f : M \to N$ be a smooth map between two manifolds and E be a vector bundle on N with a connection ∇ whose connection form is ω. Show that there is a unique connection $f^*\nabla$ on f^*E given by $f^*\omega$.

Exercise 1.22. Let (E, h) be a Hermitian vector bundle. Show that $R_i{}^{\alpha}{}_{k\bar{l}} = \sum h^{\alpha\bar{j}} R_{i\bar{j}k\bar{l}}$. In particular, $R_{k\bar{l}} = \sum R_i{}^i{}_{kl} = \sum h^{i\bar{j}} R_{i\bar{j}k\bar{l}}$

Exercise 1.23. Let (E, h) be a Hermitian vector bundle on a Hermitian manifold (M, g). We define $K_i{}^k = \sum g^{\alpha\bar{\beta}} R_i{}^k{}_{\alpha\beta}$. Show that $K_{i\bar{j}} = \sum h_{k\bar{j}} K_i{}^k$.

Exercise 1.24. Let E be a vector bundle and E^* is the dual bundle. Denote by $< \cdot, \cdot >: \mathcal{A}^p(E^*) \otimes \mathcal{A}^p(E) \to \mathcal{A}^p$ the dual pairing. Given a connection ∇ on E, we define a connection, denoted by ∇^* on E^* by $d < \xi, \eta > = < \nabla^* \xi, \eta > + < \xi, \nabla \eta >$. Show that $\nabla^* e^* = -\omega e^*$, where ω is the connection form of ∇ and $e^* = (e^1, \ldots, e^r)$ is the dual frame of a local frame $e = (e_1, \ldots, e_r)$ of E. Show that there is a naturally defined connection $\nabla^* \otimes I_r + I_r \otimes \nabla$ on $\mathcal{A}^p(E^* \otimes E)$ whose connection form is $\omega^* \otimes I_r + I_r \otimes \omega$, where I_r is the identity transformation/matrix.

Exercise 1.25. Let E and F be two vector bundles with connections ∇^E and ∇^F. Then there is a unique connection $\nabla^{E \oplus F}$ on $E \oplus F$ such that the connection matrix

$$\omega^{E \oplus F} = \begin{pmatrix} \omega^E & 0 \\ 0 & \omega^F \end{pmatrix}.$$

Exercise 1.26. Let α be a p-form with value in $\mathcal{E}nd(E)$ and ∇ be a connection with connection form ω on E. Show that $\nabla^{\mathcal{E}nd(E)} \alpha = \mathrm{d}\alpha - (\alpha \wedge \omega + \omega \wedge \alpha)$.

Exercise 1.27. Let Ω be the curvature form of a connection ∇ on a vector bundle E. Show that $\nabla\Omega = 0$, where Ω is viewed as a 2-form with values in $\mathcal{E}nd(E)$.

Exercise 1.28. Let ∇_1 and ∇_2 be two connections on E. Denote by ω_1 and ω_2 the connection forms of ∇_1 and ∇_2 respectively with respect to a frame field s. Show that $\alpha = \omega_1 - \omega_2$ transforms in the same way as the curvature form under a transformation of the frame field.

Exercise 1.29. Let (E, h) be a Hermitian vector bundle with the Hermitian connection ∇ and its curvature form Ω. Show that $(\det E, \det h)$ is an Hermitian line bundle with the curvature form $\Omega_{\det h}$. Moreover, show that $\Omega_{\det h} = \mathrm{tr}(\Omega_h)$, i.e. $R_{\alpha\bar{\beta}} = -\partial_\alpha \partial_{\bar{\beta}} \log(\det H)$.

1.3 Kähler manifolds

There are different notions of positivity in complex differential geometry. The notion of positivity in the sense of Griffiths [Gri69] is commonly used.

Definition 1.30. Let (E, h) be a holomorphic Hermitian vector bundle with curvature Ω on a complex manifold M. We say that (E, h) is of *negative*, *seminegative*, *positive*, or *semipositive* curvature in the Griffiths sense if for any $x \in M$, any $v \in E_x$, and any $\xi \in T_x^{1,0}$, we have $R(v, \bar{v}, \xi, \bar{\xi}) < 0$, ≤ 0, > 0 or ≥ 0 respectively.

Definition 1.31. A Hermitian manifold (M, g) is called Kähler, if the Kähler form ω_g is closed, i.e. $d\omega_g = 0$.

Kähler manifolds have many very nice properties (see Proposition 1, Chapter 2, [Mok89]).

Proposition 1.32. *Let (M, g) be a Hermitian manifold such that g is given by the Hermitian matrix $\left(g_{i\bar{j}}\right)$ in local coordinates (z_j). Then (M, g) is Kähler if and only if one of the following equivalent conditions holds:*

1. *types of complexified tangent vectors are preserved under parallel transport;*

2. *for any parallel real vector field v along a smooth curve γ, Jv is also parallel;*
3. $\nabla J = 0$*;*
4. $\nabla \omega = 0$*;*
5. $\mathrm{d}\omega = 0$*;*
6. *locally there exists a potential function φ such that $g_{i\bar{j}} = \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}$;*
7. *at every point $x \in M$ there exists complex geodesic coordinates (z_i) in the sense that the Hermitian metric g is represented by the Hermitian matrix $(g_{i\bar{j}})$ satisfying $g_{i\bar{j}}(x) = \delta_{ij}$ and $(\mathrm{d}g_{i\bar{j}})(x) = 0$.*

Definition 1.33. Let (E, h) be a Hermitian vector bundle. We define the Kähler form associated to h as $\omega_h = \frac{\sqrt{-1}}{2} \sum h_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$.

The following definition was introduced by Kobayashi in [Kob80] (see also [Kob87]). It has very close relation with Bogomolov slope stability of vector bundles.

Definition 1.34. A holomorphic Hermitian vector bundle (E, h) on a Hermitian manifold (M, g) is called a Hermitian-Einstein vector bundle if

$$\mathrm{Ric}_h = \lambda \omega_h$$

for some $\lambda \in \mathbb{R}$, where Ric_h is the Ricc form of the Hermitian metric h with respect to g and ω_h is the Kähler form associated to h.

Definition 1.35. A Kähler manifold (M, g) is called Kähler-Einstein if the holomorphic tangent bundle T_M is Hermitian-Einstein.

Definition 1.36. Let (M, g) be a Kähler manifold, $x \in M$ be a point and $\xi, \eta \in T_M^{1,0}$. Write $u = 2Re\xi$ and $v = 2Re\eta$. We define the holomorphic sectional curvature in the direction ξ to be

$$\mathrm{Sec} = \frac{R(u, \sqrt{-1}u; \sqrt{-1}u, u)}{\|u\|^4}$$

the holomorphic sectional curvature in the direction ξ, where R is the Riemannian curvature tensor of g. We define the holomorphic bisectional curvature in the direction (ξ, η) to be

$$\mathrm{Bisec} = \frac{R(u, \sqrt{-1}u; \sqrt{-1}v, v)}{\|u\|^2 \|v\|^2}.$$

Definition 1.37. A Kähler manifold (M, g) is said to be of constant holomorphic sectional curvature if there exists a constant λ such that in any local coordinate of M

$$R_{i\bar{j}k\bar{l}} = \lambda(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}).$$

Exercise 1.38. Show that $\mathbb{C}^n$, $\mathbb{P}^n$ and $\mathbb{B}^n$ are Kähler manifolds with constant bisectional curvature.

In fact, we have the following characterization of Kähler manifolds with constant sectional curvature. It was first proved in local version by Bochner [Boc47]. Global versions were proved later independently by Hawley [Haw53] and Igusa [Igu54]. One can find proofs in [KN96] and [Tia00].

Recall that a geodesic is a curve γ whose tangent vector remains parallel when transported along the curve, i.e. $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. With respect to the Levi-Civita connection, geodesics are unique and minimize the distance between two points. See [KN96] Section III.6 for more discussions.

Theorem 1.39 (Uniformization Theorem)**.** *If (M, g) is a complete Kähler manifold of constant sectional curvature, i.e. $R_{i\bar{j}k\bar{l}} = \lambda(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}})$ for some constant λ, then its universal covering $\tilde{M}$ is one of the manifolds $\mathbb{C}^n$, $\mathbb{P}^n$ and $\mathbb{B}^n$.*

Proof. It is enough to prove more generally that two complete simply connected n-dimensional Kähler manifolds of constant sectional curvature are biholomorphically isometric. Let M and $\tilde{M}$ be two such manifolds and choose a point $p \in M$ and $\tilde{p} \in \tilde{M}$. The completeness is required to guarantee the existence of geodesics.

After scaling, we may assume that $\lambda = -1, 0, 1$. We first prove the cases that $\lambda \leq 0$.

Consider the maps $\exp_p : T_p(M) \to M$ and $\exp_{\tilde{p}} : T_{\tilde{p}}(\tilde{M}) \to \tilde{M}$, where $\exp_p$ and $\exp_{\tilde{p}}$ are the exponential maps of the metrics g_M and $g_{\tilde{M}}$ respectively, i.e. for any $v \in T_p(M)$, $\exp_p(v)$ is the unique geodesic γ of g_M at 1 satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. It is clear that $\exp_p$ is globally defined.

Let $i : T_p(M) \to T_{\tilde p}(\tilde M)$ be a linear isometry. We define a map

$$f(q) = \exp_{\tilde p} \circ\, i \circ \exp_p^{-1}(q), \qquad q \in M.$$

To prove the theorem, it suffices to prove that $\mathrm{d}f_p = i$. In fact, assuming this claim, the rest of the proof relies on two easy facts (see [KN96] Theorem 7.9, Chapter IX): one can glue local isometric maps together in our situation; the map f is indeed holomorphic under the assumption of Kähler.

By identifying $T_v T_p M$ with $T_p M$ and $T_{i(v)} T_{\tilde p} \tilde M$ with $T_{\tilde p}\tilde M$, we have the following commutative diagram

$$\begin{array}{ccc} T_v T_p M = T_p M & \xrightarrow{\ i\ } & T_{i(v)} T_{\tilde p} \tilde M = T_{\tilde p} \tilde M \\ {\scriptstyle \mathrm{d}\exp_p(v)} \downarrow & & \downarrow {\scriptstyle \mathrm{d}\exp_{\tilde p}(i(v))} \\ T_p M & \xrightarrow{\ \mathrm{d}f_p\ } & T_{\tilde p}\tilde M. \end{array}$$

To show that $\mathrm{d}f_p = i$, it suffices to show that

$$|\mathrm{d}\exp_p(v)(w)| = |\mathrm{d}\exp_{\tilde p}(i(v))(i(w))|.$$

The essence of the proof is Cartan's Theorem (see for example P.157 [dC92]).

For any $w \in T_v(T_p M) = T_p M$, consider the parametrized surface $\gamma(s,t) = \exp_p(s(v + tw))$. We have

$$\mathrm{d}\exp_p(u)(sw) = \frac{\partial}{\partial t}\gamma(s,t),$$

where $u = s(v+tw)$. Since $\gamma(s,t)$ is geodesic for any fixed t, then $\nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial s} = 0$. Hence, $\nabla_{\frac{\partial \gamma}{\partial t}} \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial s} = 0$. Putting $X_w(s) := \frac{\partial \gamma}{\partial t}(s, 0)$. It follows that

$$\begin{aligned} 0 &= \nabla_{X_w} \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial s} \\ &= \nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{X_w} \frac{\partial \gamma}{\partial s} + \nabla_{[X_w, \frac{\partial \gamma}{\partial s}]} \frac{\partial \gamma}{\partial s} - R(\frac{\partial \gamma}{\partial s}, X_w)\frac{\partial \gamma}{\partial s} \\ &= \nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{\frac{\partial \gamma}{\partial s}} X_w - R(\frac{\partial \gamma}{\partial s}, X_w)\frac{\partial \gamma}{\partial s}. \qquad \text{(Jacobi equation)} \end{aligned}$$

A vector field $X_w(s)$ along γ satisfying the Jacobi equation is called a Jacobi field.

Now fix an orthonormal basis $e_1, \dots, e_{2n}$ for $T_p(M)$ such that $e_1 = \frac{\partial \gamma}{\partial s} |\frac{\partial \gamma}{\partial s}|^{-1}$ and $e_{n+i} = \sqrt{-1} e_i$. Let $e_i(s)$ be the parallel transport along γ, i.e. $\nabla_{\frac{\partial \gamma}{\partial s}} e_i(s) = 0$ and $e_i(0) = e_i$. Writing

$$X_w(s) = \sum X^i(s) e_i(s)$$

and using the Jacobi equation, we obtain that

$$\frac{\partial^2 X^i}{\partial s^2} e_i - R(\frac{\partial \gamma}{\partial s}, e_j) \frac{\partial \gamma}{\partial s} X^j = 0.$$

Taking inner product with e_i, we get that

$$\frac{\partial^2 X^i}{\partial s^2} - < R(\frac{\partial \gamma}{\partial s}, e_j) \frac{\partial \gamma}{\partial s}, e_i > X^j = 0$$

which is equivalent to

$$\frac{\partial^2 X^i}{\partial s^2} - |\frac{\partial \gamma}{\partial s}|^2 R(e_1, e_j, e_1, e_i) X^j = 0.$$

Therefore, X_w is uniquely determined by w and $R(e_1, e_j, e_1, e_i)$ which is uniquely determined by the holomorphic sectional curvature. Similarly, if $\tilde{X}_{i(w)}$ is the Jacobi field along the corresponding geodesic $\tilde{\gamma}$, i.e. $\tilde{\gamma}(s,t) = \exp_{\tilde{p}}(s(i(v) + ti(w)))$, then $\tilde{X}_{i(w)}$ satisfies the same Jacobi equation. Therefore, $|X_w| = |X_{i(w)}|$.

If $\lambda > 0$, then the Ricc curvature is also positive and $\tilde{M}$ is compact by Myers' Theorem. We can choose a open dense subset $U \subset M$ and show in the same way as above that U is isometry to $f(U)$. Since $\tilde{M}$ is compact, we can extend f to all $\mathbb{CP}^n$ so that f remains isometry. Q.E.D

Remark 1.40. By Hopf-Rinow Theorem (see Chapter 7, Section 2, [dC92]), we know that a connected compact manifold is complete.

Exercise 1.41. Show that on a Kähler manifolds, the curvature form satisfies the following equalities.

$$R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}} = R_{i\bar{l}k\bar{j}},$$

$$R_{i\bar{j}k\bar{l}} = R_{k\bar{l}i\bar{j}}.$$

In particular, we have $R_{i\bar{j}} = K_{i\bar{j}}$.

2 Chern classes

There are many equivalent definitions of Chern classes of vector bundles. We start by defining Chern classes using Hirzebruch's axiomatic definition. Then we will give the geometric definition using curvature.

Definition 2.1. For each rank r complex vector bundle E on a complex manifold M and each integer $i \geq 0$, the i-th Chern class $c_i(E)$ is an element in $H^{2i}(M, \mathbb{R})$ with $c_0(E) = 1$ satisfying the following axioms:

C1 (Naturality) If $f : M' \to M$ is a differentiable map. Then $c_i(f^*E) = f^*c_i(E)$.

C2 (Whitney sum formula) Let F be another complex vector bundle on M. Then $c(E \oplus F) = c(E)c(F)$, where $c(E) = \sum_{i=0} c_i(E)$ is called the total Chern class of E.

C3 (Normalization) Let L be the tautological line bundles on the projective space $\mathbb{P}^n$. Then $c(L) = 1 - h$, where h is the generator of $H^2(\mathbb{P}^n, \mathbb{Z})$. i.e. h is the Poincáre dual of the hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$.

One useful tool to use this definition to calculate Chern classes is the splitting principle.

Theorem 2.2 (Splitting principle)**.** *Let E be a complex vector bundle on M. Then there exists a complex manifold N and a morphism $f : N \to M$ such that $f^* : H^*(M) \to H^*(N)$ is an injective morphism and f^*E splits into direct sum of line bundles, i.e. $f^*E = L_1 \oplus L_2 \oplus \cdots \oplus L_r$.*

Using splitting principle and naturality, we see that total Chern class $c(E) = c(L_1) \dots c(L_r)$. The first Chern classes of the line bundles $c_1(L_i)$ are called Chern root of E. In particular, $c_i(E) = 0$ for $i > r$.

Since exact sequence of sheave of smooth functions splits, then we have a generalization of Whitney sum formula, namely, for an exact sequence of vector bundles

$$0 \to F \to E \to G \to 0$$

we have $c(E) = c(F)c(G)$.

Example 2.3. On a projective space $\mathbb{P}^n$, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to T_{\mathbb{P}^n} \to 0$$

called Euler sequence. Using Whitney sum formula, we find that $c(T_{\mathbb{P}^n}) = (1+h)^{n+1}$, where h is the generator of $H^2(\mathbb{P}^n, \mathbb{R})$.

Example 2.4. Let X be a complex projective manifold and Z be a smooth sub-manifold. Then we have the following exact sequence

$$0 \to T_Z \to T_X|_Z \to N_{Z/X} \to 0,$$

where $N_{Z/X}$ is called the normal bundle. Therefore, $c_1(T_X|_Z) = c_1(T_Z) + c_1(N_{Z/X})$.

Lemma 2.5 (Adjunction formula)**.** *Let X be a complex projective manifold and H be a smooth hypersurface. Then the normal bundle $N_{H/X}$ is isomorphic to $\mathcal{O}_H(H)$.*

Proof. Consider conormal bundle which is the dual bundle $N^*_{H/X}$ of the normal bundle. The conormal bundle fits in the following exact sequence

$$0 \to N^*_{H/X} \to \Omega^1_X|_H \to \Omega^1_H \to 0.$$

It is sufficiently to show that $N^*_{H/X} \otimes \mathcal{O}_H(H)$ is trivial. To prove this, we only need to produce a everywhere nonzero global section of $N^*_{H/X} \otimes \mathcal{O}_H(H)$. Assume that H is locally defined by $f_\alpha \in \mathcal{O}_X(U_\alpha)$, then the line bundle $[H] = \mathcal{O}_X(H)$ is given by transition functions $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$. Now since $f_\alpha \equiv 0$ on $H \cap U_\alpha$ and H is smooth, then the differential $\mathrm{d}f_\alpha$ is everywhere nonzero section of $N^*_{H/X}$ over U_α. Over $U_\alpha \cap U_\beta \cap V$, we have $\mathrm{d}f_\alpha = \mathrm{d}g_{\alpha\beta}f_\beta = g_{\alpha\beta}\mathrm{d}f_\beta + f_\beta \mathrm{d}g_{\alpha\beta} = g_{\alpha\beta}\mathrm{d}f_\beta$. Therefore, the transition functions of $N^*_{H/X}$ is $\frac{1}{g_{\alpha\beta}} = g_{\beta\alpha}$. Therefore, the line bundle $N^*_{H/X} \otimes \mathcal{O}_H(H)$ is trivial. Q.E.D

From the definition, we see that Chern classes if exist must be uniquely determined. There are different ways to construct Chern classes. In complex differential geometry, there is a more explicit way using invariant polynomials of curvature forms.

Denote by $M_r(\mathbb{C})$ the algebra of $r \times r$ complex-valued matrices. A symmetric multilinear k-form on $M_r(\mathbb{C})$ is a polynomial function $f : M_r^k \to \mathbb{C}$ such that $f(X_1, \cdots, X_k)$ is linear in each variable X_i and symmetric in $X_1, \ldots, X_k \in M_r(\mathbb{C})$, i.e. $f(X_{\sigma(1)}, \cdots, X_{\sigma(k)}) = f(X_1, \cdots, X_k)$ for any element σ in the symmetric group S_k. We say that a symmetric multilinear k-form f is $GL(r, \mathbb{C})$-invariant if

$$f(aX_1a^{-1}, \cdots, aX_ka^{-1}) = f(X_1, \cdots, X_k)$$

for any $a \in GL(r, \mathbb{C})$. We write $F(X) = f(X, X, \ldots, X)$ which is a $GL(r, \mathbb{C})$-invariant polynomial. The k-form f is called a normalized (complete/full) polarization of F. One can recover f from F by the following identity

$$f(X_1, \ldots, X_k) = \frac{1}{k!}\frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} F(t_1X_1 + \cdots + t_kX_k)(0).$$

Exercise 2.6. Let f be a $GL(r, \mathbb{C})$-invariant symmetric multilinear k-form. Then we can write

$$f(X_1, \ldots, X_k) = \sum \lambda_{\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_k} x_{1,\alpha_1\beta_1} \cdots x_{k,\alpha_k\beta_k},$$

where $X_i = \left(x_{i,\alpha\beta}\right)$, $1 \le i \le k$ and $1 \le \alpha, \beta \le r$. Show that

$$f([Y, X_1], X_2, \ldots, X_k) + f(X_1, [Y, X_2], \ldots, X_k) + \cdots + f(X_1, X_2, \ldots, [Y, X_k]) = 0$$

by differentiating

$$f(e^{tY}X_1e^{-tY}, \ldots, e^{tY}X_ke^{-tY}) = f(X_1, \ldots, X_k)$$

with respect to t at $t = 0$. Assume that Y is a matrix of one-form and X_i are matrices of p_i-forms. Show that

$$\sum_{i=1}^{k} (-1)^{p_1+\cdots+p_{i-1}} f(X_1, \ldots, [Y, X_i], \ldots, X_k) = 0.$$

Theorem 2.7 (Chern-Weil)**.** *Let E be a rank r complex vector bundle on a complex manifold M and ∇ be a connection of E. Denote by Ω the curvature form associated to ∇. Given a $GL(r, \mathbb{C})$-invariant symmetric multilinear form f of degree k. The form $\gamma = f(\Omega, \ldots, \Omega)$ is a closed $2k$-form in $H^{2k}(M, \mathbb{C})$ and the cohomology class $[\gamma]$ does not depend on choices of connections.*

Proof. Since f is $GL(r,\mathbb{C})$-invariant symmetric multilinear, we have

$$f([Y,X_1],X_2,\dots,X_k)+f(X_1,[Y,X_2],\dots,X_k)+\cdots+f(X_1,X_2,\dots,[Y,X_k])=0.$$

Moreover, if Y is a matrix of one-form and X_i are matrices of p_i-forms, then

$$\sum_{i=1}^{k}(-1)^{p_1+\cdots+p_{i-1}}f(X_1,\dots,[Y,X_i],\dots,X_k)=0.$$

Now using Bianchi identity, we see that

$$\begin{aligned} \mathrm{d}f(\Omega,\dots,\Omega) &= f(\mathrm{d}\Omega,\dots,\Omega)+\cdots+f(\Omega,\dots,\mathrm{d}\Omega)\\ &= -(f([\Omega,\omega],\dots,\Omega)+\cdots+f(\Omega,\dots,[\Omega,\omega]))=0. \end{aligned}$$

We now show that $[\gamma]$ does not depend on connections. Consider two connections ∇ and ∇', and define $\nabla_t=(1-t)\nabla+t\nabla'$, where $0\le t\le 1$. It can be checked that ∇_t is a connection on M for any t. With respect to local frame fields, we have

$$\begin{aligned} \omega_t &= \omega_1+t(\omega_2-\omega_1)\\ \Omega_t &= \mathrm{d}\omega_t-\omega_t\wedge\omega_t. \end{aligned}$$

Set $\alpha=\omega_2-\omega_1$. Differentiating Ω_t with respect to t, we get

$$\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}=\mathrm{d}\alpha-(\alpha\wedge\omega_t+\omega_t\wedge\alpha)=\mathrm{d}\alpha-[\omega_t,\alpha].$$

We define

$$\varphi=k\int_0^1 f(\alpha,\Omega_t,\dots,\Omega_t)\mathrm{d}t.$$

One can check that α transforms in the same way as the curvature form under a transformation of the frame field. Hence φ is a globally defined $2k-1$ form on M. Consider the exterior derivative $\mathrm{d}f(\alpha,\Omega_t,\dots,\Omega_t)$, we see that

$$\begin{aligned} \mathrm{d}f(\alpha,\Omega_t,\dots,\Omega_t) &= f(\mathrm{d}\alpha,\Omega_t,\dots,\Omega_t)-\sum_{i=2}^{k}f(\alpha,\Omega_t,\dots,\overbrace{[\omega_t,\Omega_t]}^{i},\dots,\Omega_t)\\ &= f(\mathrm{d}\alpha,\Omega_t,\dots,\Omega_t)-f([\omega_t,\alpha],\Omega_t,\dots,\Omega_t)\\ &= f(\frac{\mathrm{d}\Omega_t}{\mathrm{d}t},\Omega_t,\dots,\Omega_t)\\ &= \frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}f(\Omega_t,\Omega_t,\dots,\Omega_t). \end{aligned}$$

Hence,

$$\mathrm{d}\varphi = k\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(\Omega_t,\ldots,\Omega_t)\mathrm{d}t = f(\Omega_2,\ldots,\Omega_2) - f(\Omega_1,\ldots,\Omega_1)$$

which proves that the cohomology class γ does not depend on connections. Q.E.D

Remark 2.8. One can also prove that $F(\Omega)$ is independent of choices of connections by showing that $F(\Omega_1)$ and $F(\Omega_2)$ are homotopy equivalence hence represents the same cohomology class in the de Rham group (see for example Lemma 18.2, [MT97]).

Theorem 2.9. *Let E be a rank r complex vector bundle on a complex manifold M with a curvature form Ω. Then the i-th Chern classes $c_i(E)$ is represented by the $2i$-form γ_i, where*

$$1+\gamma_1+\gamma_2+\cdots+\gamma_r = \det(I_r - \frac{1}{2\pi\sqrt{-1}}\Omega).$$

Proof. It is clear that $c_0(E) = 1$. Axiom 1 and 2 follows from Exercises 1.21 and 1.25. For Axiom 3, since $H^2(\mathbb{P}^n,\mathbb{C}) \to H^2(\mathbb{P}^1,\mathbb{C})$ is an isomorphism induced by the inclusion $j:\mathbb{P}^1 \subset \mathbb{P}^n$. We only need to prove the Axiom 3 for $\mathbb{P}^1$. We take the curvature

$$\Omega = -\partial\bar{\partial}\log(1+|z|^2) = -\frac{\mathrm{d}z\wedge \mathrm{d}\bar{z}}{(1+|z|^2)^2}$$

associated to the Hermitian/Chern connection of the tautological line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ over the open set $U_0 = [z_0 : z_1] \mid z_0 \neq 0$, where $z = \frac{z_1}{z_0}$. Then

$$\gamma_1 = -\frac{1}{2\pi\sqrt{-1}}\Omega = \frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{2\pi\sqrt{-1}(1+|z|^2)^2}.$$

Using polar coordinate (r,θ) define by $z = re^{2\pi\sqrt{-1}\theta}$, we write

$$\gamma_1 = \frac{-2r\mathrm{d}r\wedge\mathrm{d}\theta}{(1+r^2)^2}$$

on U. Integrating γ_1 over $\mathbb{P}^1$, we obtain

$$\int_{\mathbb{P}^1}\gamma_1 = \int_{U_0}\gamma_1 = -\int_0^{+\infty}\int_0^1 \frac{2r\mathrm{d}r\wedge\mathrm{d}\theta}{(1+r^2)^2} = -1.$$

Here we apply a partition of unity $\rho_0 + \rho_1 = 1$, where Supp (ρ_1) is concentrate at the infinity point $[0:1] \in \mathbb{P}^1$. Q.E.D

Remark 2.10. Note that the polynomials $\gamma_p(X)$ generates the algebra of $GL(r,\mathbb{C})$-invariant polynomials on $M_r(\mathbb{C})$.

Remark 2.11. A simple linear algebra calculation shows that

$$\gamma_p = \frac{(-1)^p}{(2\pi\sqrt{-1})^p p!}\sum \delta^{j_1\cdots j_p}_{i_1\cdots i_p}\Omega^{i_1}_{j_1}\wedge\cdots\wedge\Omega^{i_p}_{j_p}.$$

In particular,

$$\gamma_1 = -\frac{1}{2\pi\sqrt{-1}}\sum \Omega^j_j.$$

$$\gamma_2 = -\frac{1}{8\pi^2}\sum \Omega^j_j\wedge\Omega^k_k - \Omega^j_k\wedge\Omega^k_j.$$

Exercise 2.12. In Theorem 2.9, the number $\int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(-1))$ is called the degree $\deg\mathcal{O}_{\mathbb{P}^1}(-1)$ of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$. Show that $\deg\mathcal{O}_{\mathbb{P}^1}(k) = k$.

Definition 2.13. Let L be a line bundle. We say $c_1(L)$ is positive if there exists a metric h on L with curvature form Ω_h such that $\frac{\sqrt{-1}}{2\pi}\Omega_h$ is a positive $(1,1)$-form.

Recall that A $(1,1)$-form ω on a complex manifold M is positive if we can write $\omega = \frac{\sqrt{-1}}{2}\sum_{ij}h_{ij}\mathrm{d}z_i\wedge\mathrm{d}\bar{z}_j$ with $H = \left(h_{ij}\right)$ a positive definite matrix.

Example 2.14. Let $\mathcal{O}_{\mathbb{CP}^n}(1)$ is the dual of the tautological line bundle. Then $c_1(\mathcal{O}_{\mathbb{CP}^2}(1)) > 0$.

Theorem 2.15 (Gauss-Bonnet). *Let S be a complex projective surface. Then*

$$c_2(S) = \chi_{top}(S),$$

where χ_{top} is the topological Euler characteristic.

Let M be a complex manifold, denote by $\mathcal{O}_M$ the structure sheaf of holomorphic functions on M and $\mathcal{O}^*_M$ the sheaf of invertible holomorphic functions. One can check that the set of isomorphism classes of line bundles on M is bijective to $H^1(M,\mathcal{O}^*_M)$. On M, we have the following exact sequence

$$0\to Z\to\mathcal{O}_M\to\mathcal{O}^*_M\to 0$$

called the exponential sequence. Taking cohomology, we get the boundary map $\delta : H^1(M, \mathcal{O}_M^*) \to H^2(M, Z)$. One see that δ is “coincide” with the Chern-Weil homomorphism c_1.

Proposition 2.16. *Let L be a complex line bundle on M. Then*

$$\delta(L) = -\frac{c_1(L)}{2\pi\sqrt{-1}}.$$

Proof. Descending to differential manifold, we need to show that the diagram

$$\begin{array}{ccc} H^1(M, \mathcal{O}_M^*) & \longrightarrow & H^1(M, C_{\mathbb{C}}^*) \\ \delta\big\downarrow & \overset{\cong}{\swarrow} & \big\downarrow c_1 \\ H^2(M, \mathbb{Z}) & \longrightarrow & H^2(M, \mathbb{C}) \end{array}$$

commutes, where $C_{\mathbb{C}}^*$ is the sheaf of complex-valued differentiable functions on M. It suffice to show the lower triangle commutes. This can be done by comparing de Rham resolution and Cěch resolution of $\mathbb{C}$.

Let $M = \cup U_i$ be an trivialization of a line bundle L, i.e. $\psi_i : L|_{U_i} \cong U_i \times \mathbb{C}$. Then $\psi_{ij} = \psi_i \circ \psi_j^{-1}$ are sections in $C_{\mathbb{C}}^*(U_{ij})$ representing L. The boundary map is given by $\delta(L) = \{U_{ijk}, \frac{1}{2\pi\sqrt{-1}}(\log\psi_{ij} - \log\psi_{ik} + \log\psi_{jk})\} \in H^2(M, \mathbb{Z})$. Now choose a connection D on L. Denote by ω the connection form of D with respect to the trivialization ψ_i, i.e. $\omega_i = \psi_{ij}^{-1}\omega_j\psi_{ij} + \psi_{ij}^{-1}\mathrm{d}\psi_{ij}$. Since L is a line bundle, then $\omega_i = \omega_j + \psi_{ij}^{-1}\mathrm{d}\psi_{ij}$. Equivalently, $\omega_i - \omega_j = \mathrm{d}(\log\psi_{ij})$. The curvature form $\Omega = \mathrm{d}\omega + \omega \wedge \omega = \mathrm{d}\omega$, because ω is simply a 1×1 matrix of 1-form. We want to see which Cěch 2-cocycle represents Ω Now look at the de Rham resolution of $\mathbb{C}$:

$$0 \to \mathbb{C} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \cdots.$$

Locally, $(U_i, \Omega_i) = (U_i, \mathrm{d}\omega_i)$. So it can be represented by a 1-form (U_i, ω_i) in $\mathcal{A}^1(U_i)$. Take the Cěch boundary map $\delta_1 : \mathcal{A}^1(U_i) \to \mathcal{A}^1(U_ij)$, we get $\delta_1(\omega_i) = \omega_j - \omega_i \in \mathcal{A}^1(U_{ij})$. So we can use $\omega_j - \omega_i = -\mathrm{d}(\log\psi_{ij})$ to represent (U_i, Ω_i). Unapply d, we get an element $\log(\psi_{ij})$ in $\mathcal{A}^0(U_{ij})$. Again apply Cěch boundary map $\delta_2 : \mathcal{A}^0(U_{ij}) \to \mathcal{A}^0(U_{ijk})$, we see that Ω_i can be represented by $-(\log(\psi_{jk}) - \log(\psi_{ik}) + \log(\psi_{ij}))$. Therefore, $c_1(L) = -\delta(L)2\pi\sqrt{-1}$.

Q.E.D

3 Miyaoka-Yau inequality

In this section we will prove the Miyaoka-Yau inequality on algebraic surfaces. It was proved independently by Yau [Yau77, Yau78] and Yoichi Miyaoka [Miy77], after weaker versions proved by Bogomolov [Bog78] and Van de Ven [VdV66]. For proofs of the inequality using algebraic method, one can also consult the books [BHPVdV04].

Definition 3.1. A smooth surface S is called a ball quotient surface if it is bilohomorphic to the quotient space $\mathbb{B}/\Gamma$, where $\mathbb{B} \subset \mathbb{C}^2$ is the unit ball and Γ is a discrete group.

Using differential geometry methods, Yau prove furthermore that $c_1^2 = 3c_2$ implies the surface is a quotient of the unit ball. The proof uses the following celebrating theorem.

Theorem 3.2 (Aubin-Yau)**.** *Let M be a compact Kähler manifold such that the first Chern class $c_1(M)$ is negative. Then M is Kähler-Einstein.*

Using Aubin-Yau Theorem we can prove the Miyaoka-Yau inequality. In fact, assuming M is Kähler-Einstein, then inequality was known much earlier.

Theorem 3.3 (Miyaoka-Yau)**.** *Let S be a compact Kähler surface such that $c_1(S)$ is negative. Then $c_1^2(S) \leq 3c_2(S)$ and the equality hold if and only if S is a quotient of ball.*

Proof. By Aubin-Yau Theorem, S has a Kähler-Einstein metric h. Choose a unitary local frame of T_S, i.e. a frame (e_1^*, e_2^*) such that $h(e_i^*, \bar{e}_j^*) = \delta_{ij}$. Denote by (e_1, e_2) the coframe of (e_1^*, e_2^*). Under this unitary coframe, we can write $\omega_h = \frac{\sqrt{-1}}{2}(e_1 \wedge \bar{e}_1 + e_2 \wedge \bar{e}_2)$ and $e_k \wedge \bar{e}_k \wedge e_l \wedge \bar{e}_l = -2\omega_h^2$ for $1 \leq k \neq l \leq 2$. And we can write $\Omega_{kl} = \dfrac{1}{2}\sum_{i,j=1}^{2} R_{i\bar{j}k\bar{l}} e_i \wedge \bar{e}_j$.Recall that on a Kähler manifold, we have the following identities:

$$R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}} = R_{i\bar{l}k\bar{j}},$$

$$R_{i\bar{j}k\bar{l}} = R_{k\bar{l}i\bar{j}}.$$

Set $r_{i\bar{j}} = \sum_{k=1}^{2} R_{i\bar{j}k\bar{k}}$ and $s = \sum_{i=1}^{2} r_{i\bar{i}}$. Then the Ricci form $\mathrm{Ric}_h = Tr\Omega = \sum_{i,j=1}^{2} r_{i\bar{j}} e_i \wedge \bar{e}_j$. Notice that $r_{i\bar{j}} = \overline{r_{j\bar{i}}}$. Hence s is real. We will use the following notations

$$|r|^2 = \sum_{i,j=1}^{2} |r_{i\bar{j}}|^2, \quad |R|^2 = \sum_{i,j,k,l=1}^{2} |R_{k\bar{l}i\bar{j}}|^2.$$

Since S is Kähler-Einstein, under the unitary fram, we see that $r_{i\bar{j}} = 0$ for $i \neq j$ and $r_{1\bar{1}} = r_{2\bar{2}}$ from the identity $Tr\Omega = \lambda\omega_h$. Therefore, $s^2 = 2|r|^2 = 2\lambda^2$ is a constant number. The first and second Chern classes are defined by

$$c_1(S) = -\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{2} \Omega_{ii}, \quad c_2(S) = (\frac{1}{2\pi\sqrt{-1}})^2 \frac{1}{2} \sum_{i,j=1}^{2} (\Omega_{ii}\Omega_{jj} - \Omega_{ij}\Omega_{ji}).$$

Therefore, we have

$$8\pi^2(c_1^2 - 3c_2) = \sum_{i,j=1}^{2} (\Omega_{ii}\Omega_{jj} - 3\Omega_{ij}\Omega_{ji})$$

Since $e_i \wedge \bar{e}_j \wedge e_k \wedge \bar{e}_l = 0$ unless $i = j \neq k = l$ or $i = l \neq j = k$, in which case, it equals to $\pm 2\omega_h^2$, we have

$$\begin{aligned}
4\sum_{i,j=1}^{2} \Omega_{ii}\Omega_{jj} &= \sum_{1\le k\neq l\le 2} \sum_{i,j=1}^{2} (R_{k\bar{k}i\bar{i}}R_{l\bar{l}j\bar{j}} - R_{k\bar{l}i\bar{i}}R_{l\bar{k}j\bar{j}}) e_k \wedge \bar{e}_k \wedge e_l \wedge \bar{e}_l \\
&= \sum_{k,l=1}^{2} (r_{k\bar{k}}r_{l\bar{l}} - r_{k\bar{l}}r_{l\bar{k}}) e_k \wedge \bar{e}_k \wedge e_l \wedge \bar{e}_l \\
&= -2(s^2 - |r|^2)\omega_h^2
\end{aligned}$$

$$\begin{aligned}
4\sum_{i,j=1}^{2} \Omega_{ij}\Omega_{ji} &= \sum_{1\le k\neq l\le 2} \sum_{i,j=1}^{2} (R_{k\bar{k}i\bar{j}}R_{l\bar{l}j\bar{i}} - R_{k\bar{l}i\bar{j}}R_{l\bar{k}j\bar{i}}) e_k \wedge \bar{e}_k \wedge e_l \wedge \bar{e}_l \\
&= \sum_{k,l=1}^{2} (r_{k\bar{l}}r_{l\bar{k}} - \sum_{i,j=1}^{2} |R_{k\bar{l}i\bar{j}}|^2) e_k \wedge \bar{e}_k \wedge e_l \wedge \bar{e}_l \\
&= -2(|r|^2 - |R|^2)\omega_h^2.
\end{aligned}$$

Here we use the skew-Hermitian property, i.e. $\overline{R_{k\bar{l}i\bar{j}}} = R_{l\bar{k}j\bar{i}}$. We have

$$16\pi^2(c_1^2 - 3c_2) = -(s^2 + 3|R|^2 - 4|r|^2)\omega_h^2 = (s^2 - 3|R|^2)\omega_h^2.$$

We claim that $s^2 \leq 3|R|^2$. Set

$$T_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \frac{s}{6}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}).$$

Then

$$|T|^2 = \sum_{i,j,k,l=1}^{2} |T_{i\bar{j}k\bar{l}}|^2 = |R|^2 - \frac{s^2}{3} \geq 0.$$

When $|T|^2 = 0$, then $T = 0$ which is equivalent to say that the curvature R induces a constant holomorphic sectional curvature (see Proposition 7.6, [KN96]). Furthermore, the constant is negative since $c_1(S)$ is negative which implies that S is covered be the unit ball (see for example Theorem 7.9, [KN96] or Theorem 1.12, [Tia00]).

Q.E.D

Exercise 3.4. Let S be a general (hence smooth by Bertini's theorem) degree 5 surface in $\mathbb{CP}^3$. Find $c_1(S)$ and $c_2(S)$. Show that $c_1^2(S) < 3c_2(S)$.

4 Preliminaries on complex algebraic surfaces

In this section, we will briefly review intersection theory on algebraic surfaces and resolution of singularities. A very good reference is the book Complex Algebraic Surfaces by Arnaud Beauville [Bea96].

4.1 Intersection theory on algebraic surfaces

On complex algebraic surfaces, there are deferent ways to define intersection numbers of curves on surfaces. However, they are all the same if the certain nice properties (naturality, commutativity, associativity in Theorem 4.3) are satisfied.

Definition 4.1. Let C_1 and C_2 be two distinct irreducible curves on a smooth projective surface S. The intersection multiplicity of C_1 and C_2 at a point $p \in S$ is defined by

$$I_p(C_1, C_2) = \dim_k \mathcal{O}_{S,p}/(f, g),$$

where f and g are defining equations of C_1 and C_2 in $\mathcal{O}_{S,p}$. The intersection number of C_1 and C_2 is defined by

$$C_1 \cdot C_2 = \sum_{p \in C_1 \cap C_2} I_p(C_1, C_2).$$

Let C_1 and C_2 be two curves on S. We say that C_1 and C_2 intersect transversally at a point p if the local defining equations f and g generates the maximal ideal $\mathfrak{m}_p$ of the local ring $\mathcal{O}_{S,p}$. Equivalently, $I_p(C_1, C_2) = 1$. It is also equivalent to that $C_1 \cup C_2$ at p is defined by $xy = 0$ for a suitable choice of local coordinate system.

Example 4.2. Let C_1 be the curve $y - x^2 = 0$ and C_2 be the line $y = 0$. The intersection of C_1 and C_2 at the origin $o \in \mathbb{C}^2$ is 2.

Theorem 4.3. *Let S be a smooth projective surface. Then there exists a unique bilinear paring*

$$\begin{aligned} \operatorname{Pic}(S) \times \operatorname{Pic}(S) &\to \mathbb{Z} \\ (C, D) \qquad &\mapsto C \cdot D \end{aligned}$$

such that

(a) Naturality: If C and D are smooth curves intersect transversally, then $C \cdot D = \#(C \cap D)$;
(b) Commutativity: $C \cdot D = D \cdot C$;
(c) Associativity: $(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D$.

Proof. Define $C \cdot D = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) - \chi(\mathcal{O}_S(-D)) + \chi(\mathcal{O}_S(-C-D))$. Then it is well defined, i.e. $C \cdot D = 0$ if $\mathcal{O}_S(C) \cong \mathcal{O}_S$. The natuality and commutativity follow from the exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_S(-C-D) \to &\mathcal{O}_S(-C) \oplus \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_{C \cap D} \to 0 \\ &\qquad\qquad (u, v) \mapsto us + vt \\ w \mapsto &(wt, -ws) \end{aligned}$$

where $s \in H^0(S, \mathcal{O}_S(C))$ and $t \in H^0(S, \mathcal{O}_S(D))$ are non-zero sections vanishing on C and D respectively. To show the associativity, we need the following two lemmas.

Lemma 4.4. *Let C be a smooth curve on a smooth projective surface S and D be any divisor on S. Then*

$$C \cdot D = \deg \mathcal{O}_C(D).$$

Proof. From the following exact sequence

$$0 \to \mathcal{O}_S(-C-D) \to \mathcal{O}_S(-D) \to \mathcal{O}_C(-D) \to 0,$$

we see that $\chi(\mathcal{O}_C(-D)) = \chi(\mathcal{O}_S(-D)) - \chi(\mathcal{O}_S(-C-D))$. Applying Riemann-Roch to $\mathcal{O}_C(-D)$, we know that $\chi(\mathcal{O}_C(-D)) = -deg\mathcal{O}_C(D) + \chi(\mathcal{O}_C) = -deg\mathcal{O}_C(D) + (\chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)))$. Therefore, $C \cdot D = deg\mathcal{O}_C(D)$. Q.E.D

Lemma 4.5 (Moving Lemma)**.** *Let D be a divisor on a smooth projective surface. Then there exist very ample divisors A and B on S such that $D = A - B$. Moreover, by Bertini's theorem, we may assume that A and B are smooth.*

Proof. Let L be an ample line bundle. Then by Theorem 4.27 and Theorem 4.24, there exists a integer $m > 0$ such that L^m and $L^m \otimes D$ are both very ample. By Bertini's theorem, there exists smooth divisors A and B linearly equivalent to L^m and $L^m \otimes D$ respectively. Then D is linearly equivalent to $A - B$. Q.E.D

We write $s(C_1, C_2, C_3) = C_1 \cdot (C_2 + C_3) - C_1 \cdot C_2 - C_1 \cdot C_3$ and prove that $s(C_1, C_2, C_3) = 0$. By definition

$$\begin{aligned} s(C_1, C_2, C_3) = &-\chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(-C_1)) + \chi(\mathcal{O}_S(-C_2)) + \chi(\mathcal{O}_S(-C_3)) \\ &- \chi(\mathcal{O}_S(-C_2 - C_3)) - \chi(\mathcal{O}_S(-C_1 - C_2)) - \chi(\mathcal{O}_S(-C_1 - C_3)) \\ &+ \chi(\mathcal{O}_S(-C_1 - C_2 - C_3)). \end{aligned}$$

Then $s(C_1, C_2, C_3)$ is symmetric in C_1, C_2 and C_3. Assume that one of C_1, C_2 and C_3 is smooth. Using Lemma 4.4, we see that $s(C_1, C_2, C_3) = 0$. In general, we may write $C_2 = A - B$ where A and B a very ample smooth curves. Then $s(C_1, C_2, B) = 0$. It follows that $C_1 \cdot (C_2 + B) = C_1 \cdot C_2 + C_1 \cdot B$. Equivalently, $C_1 \cdot (A - B) = C_1 \cdot A - C_1 \cdot B$. Now write $C_1 = M - N$, where M and N are very ample smooth curves. Now using commutativity, we conclude that

$$\begin{aligned}C_1 \cdot (C_2 + C_3) =& A \cdot (C_2 + C_3) - B \cdot (C_2 + C_3)\\ =& A \cdot C_2 - B \cdot C_2 + A \cdot C_3 - B \cdot C_3\\ =& (A - B) \cdot C_2 + (A - B) \cdot C_3\\ =& C_1 \cdot C_2 + C_1 \cdot C_3.\end{aligned}$$

The uniqueness follows from the moving lemma and property (a). Q.E.D

From the proof of the theorem, we define intersection numbers algebraically.

Definition 4.6. The intersection number of two divisors C and D on a smooth projective surface S is defined as

$$C \cdot D = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) - \chi(\mathcal{O}_S(-D)) + \chi(\mathcal{O}_S(-C-D)).$$

Remark 4.7. Let S be a compact complex surface. We have the following bilinear cup product paring

$$\begin{aligned}H^2(S,\mathbb{Z}) \times H^2(S,\mathbb{Z}) &\to H^4(S,\mathbb{Z}) = \mathbb{Z}\\ (\xi, \eta) &\mapsto (\xi, \eta).\end{aligned}$$

One can show that the cup product also satisfies the properties in Theorem 4.3. In fact, only Property (1) need to be verified. Using Poincaré duality, one can prove that

$$(c_1(\mathcal{O}_S(C)), c_1(\mathcal{O}_S(D))) = \int_C c_1(\mathcal{O}_C(D)) = \#(C \cap D).$$

Via the boundary map $\mathrm{Pic}(S) = H^1(S, \mathcal{O}_S) \to H^2(S, \mathbb{C})$, we see that

$$C \cdot D = (c_1(\mathcal{O}_S(C)), c_1(\mathcal{O}_S(D))) = \int_S c_1(\mathcal{O}_S(C)) \wedge c_1(\mathcal{O}_S(D))$$

by the uniqueness of intersection form.

Different definitions of the same object have different advantages. For example, using the algebraic definition of intersection number, we can easily get the Riemann-Roch formula on smooth algebraic surfaces.

Theorem 4.8 (Riemann-Roch)**.** *Let S be a smooth projective surface and D a divisor on S. Then*

$$\chi(\mathcal{O}_S(D)) = \frac{D(D-K_S)}{2} + \chi(\mathcal{O}_S).$$

Proof. By Serre duality and the (algebraic) definition of intersection number, we get the following equalities from which the Riemann-Roch formula follows.

$$\begin{aligned} D(D-K_S) &= -(-D)(D-K) \\ &= -\chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(D)) + \chi(\mathcal{O}_S(D-K)) - \chi(\mathcal{O}_S(K_S)) \\ &= -2\chi(\mathcal{O}_S) + 2\chi(\mathcal{O}_S(D)). \end{aligned}$$

Q.E.D

Intersection theory provides numerical tools to study algebraic surface. We introduce a very important concept called numerical equivalence.

Definition 4.9. Let S be a smooth projective surface

1. A divisor D on S is numerically trivial, denoted by $D \equiv 0$ if $C \cdot D = 0$ for any irreducible curve C in S.
2. Two divisors D_1 and D_2 are numerically equivalent if $D_1 - D_2 \equiv 0$.
3. A divisor D is said nef (numerically effective) if $C \cdot D \geq 0$ for any irreducible curve C.
4. We denote $N^1(S)$ the group of numerically equivalent classes of divisors on S.

Proposition 4.10. *Let $f : S \to C$ be a surjective morphism from a smooth projective surface to a smooth projective curve. Denote by F a general fiber of f. Then $F^2 = 0$.*

Proof. By moving lemma $F = f^*p = f^*A - f^*B$, where A and B are very ample divisors on C such that $p \notin A \cup B$. Then $F \cdot F = (f^*A - f^*B) \cdot f^*p = 0 - 0 = 0$. Q.E.D

Proposition 4.11. *Let S be a smooth projective surface.*

1. *Let C_1 and C_2 be two distinct irreducible curves on S. Then $C_1 \cdot C_2 \geq 0$.*

2. *Let D be an effective divisor and C be an irreducible curve on S. If $C \nsubseteq \text{Supp}(D)$, then $C \cdot D \geq 0$. If $C \cdot D < 0$, then $C \subseteq \text{Supp}(D)$ and $C^2 < 0$.*

Proof. The conclusions follow from the definition of intersection form. Q.E.D

Proposition 4.12. *Let $f : S_1 \to S_2$ be a generically finite morphism of degree d. Then $f^*C \cdot f^*D = d(C \cdot D)$.*

Proof. By moving lemma, we may assume that C and D are smooth curves meeting transversally and $C \cap D \subset S_2 - B$, where B is the locus where the set $f^{-1}(p)$ for any $p \in B$ consists of at most $d-1$ points or is a curve. Then $f^*C \cdot f^*D$ equals the number of points in $f^{-1}(C \cap D) = d(C \cdot D)$. Q.E.D

Theorem 4.13 (Bézout)**.** *Let C and D be two curves in $\mathbb{P}^2$ of degree c and d respectively. Then $C \cdot D = cd$.*

4.2 Blowing-up

Definition 4.14. The blowing-up of the affine space $\mathbb{A}^n$ at the origin 0 is the subvariety $Bl_0(\mathbb{A}^n)$ in $\mathbb{A}^n \times \mathbb{P}^{n-1}$ define by equations $x_i y_j = x_j y_i$, $i, j = 1, 2, \ldots, n$, where $(x_1, x_2, \ldots, x_n)$ and $[y_1, y_2, \ldots, y_n]$ are the coordinate systems of $\mathbb{A}^n$ and $\mathbb{P}^{n-1}$ respectively.

Theorem 4.15. *The restriction of the projection morphism $\varphi : Bl_0(\mathbb{A}^n) \to \mathbb{A}^n$ has the following properties:*

1. *$\varphi : Bl_0((\mathbb{A}^n)) - \varphi^{-1}(0) \to \mathbb{A}^n - \{0\}$ is an isomorphism;*
2. *$\varphi^{-1}(0) \cong \mathbb{P}^{n-1}$.*

Definition 4.16. Let $X \subset \mathbb{A}^n$ be a variety containing the origin 0 and $\dim Y > 0$. The blowing-up of Y at 0 is defined as the closure $\tilde{Y} = Bl_0(Y) = \overline{\varphi^{-1}(Y - \{0\})} \subset Bl_0(\mathbb{A}^n)$, where $\varphi : Bl_0(\mathbb{A}^n) \to \mathbb{A}^n$ is the blowing-up of $\mathbb{A}^n$ at the origin. To blow up at any point other point $P \in \mathbb{A}^n$, make a linear transformation sending P to 0.

Definition 4.17. Let H be a hypersurface in X passing through a point P and $\varphi : \tilde{X} \to X$ be the blowing-up of X at P. We call φ^*H the total transform of H and $\overline{\varphi^{-1}(H - \{P\})}$ the strict transform.

Example 4.18. The $Bl_0(\mathbb{P}^2)$, where $0 = [0,0,1]$, is the subvariety in $\mathbb{P}^2 \times \mathbb{P}^1$ defined by the bi-homogeneous polynomial $xt = ys$, where $[x,y,z]$ and $[s,t]$ are the homogeneous coordinates of $\mathbb{P}^2$ and $\mathbb{P}^1$ respectively. Let C be the curve in $\mathbb{P}^2$ defined $xz - y^2 = 0$. Then the total transform φ^*C is the subset in $Bl_0(\mathbb{P}^2)$ defined by $xt = ys$ and $xz - y^2 = 0$. It has two irreducible components: the strict tranform $\tilde{C}$ and a divisor E called the exceptional divisor.

Theorem 4.19. *Let S be a smooth projective surface and $p \in S$ be a point. Then there exist a smooth projective surface $\tilde{S}$ and a morphism $\varphi : \tilde{S} \to S$ which are unique up to isomorphism such that*

1. *the restriction $\varphi : \tilde{S} - \varphi^{-1}(p) \to S - \{p\}$ is an isomorphism, and*
2. *$E = \varphi^{-1}(p)$ is isomorphism to $\mathbb{P}^1$. We call E the exceptional divisor.*

Lemma 4.20. *Let C be an irreducible curve on a smooth projective surface S and $p \in S$ be a point. Assume that the multiplicity of C at p is m and $\pi : \tilde{S} \to S$ is the blowing up of S at p. Then $\pi^*C = \tilde{C} + mE$, where $\tilde{C}$ is the strict transform of C and E is the exceptional divisor.*

Proof. In any case, we know that $\pi^*C = \tilde{C} + kE$. We will show that $k = m$. Take an open neighborhood U of p in S with coordinates x and y. Since C is irreducible, we may assume that C is defined by an irreducible function $f(x,y)$ in U. By definition of multiplicity, we see that $f(x,y) = f_m(x,y) + f_{m+1}(x,y) + \cdots$. Let $((x,y),[s,t])$ be the coordinate of the blowing up. Then in the neighborhood $\tilde{U}$ of $(p,[1,0])$, the curve π^*C is defined by

$$\begin{aligned} f(x,tx) &= f_m(x,tx) + f_{m+1}(x,tx) + \cdots \\ &= x^m(f_m(1,t) + xf_{m+1}(1,t) + \cdots). \end{aligned}$$

We note that E is defined by $x = 0$. Therefore, $\pi^*C = \tilde{C} + mE$. Q.E.D

Theorem 4.21. *Let $\pi : \tilde{S} \to S$ be the blowing-up of S at a smooth point p and E be the exceptional divisor. Then*

1. *There is an isomorphism* $\mathrm{Pic}S \oplus \mathbb{Z}E \to \mathrm{Pic}\tilde{S}$ *defined by* $(C, m) \mapsto \pi^*C + mE$.
2. *Let* C *and* D *be two divisors on* S. *Then* $\pi^*C \cdot \pi^*D = C \cdot D$, $\pi^*C \cdot E = 0$ *and* $E^2 = -1$.
3. $K_{\tilde{S}} = \pi^*K_S + E$, *where* K_S *and* $K_{\tilde{S}}$ *are the canonical divisors.*

Proof.

1. Take an irreducible curve B on $\tilde{S}$. If $B \neq E$, then $B = \pi^*(\pi(B))$. Therefore $\mathrm{Pic}S \oplus \mathbb{Z}E \to \mathrm{Pic}\tilde{S}$ is surjective. If $\pi^*D + mE$ is linearly equivalent to 0, then $m = 0$ and hence π^*D is linearly equivalent to 0. It follows that $D = \pi(\pi^*D)$ is linearly equivalent to 0.
2. Moving lemma plus the previous lemma.
3. Adjunction formula $-2 = \deg(K_{\tilde{S}} + E)|_E = (\pi^*K_S - kE + E) \cdot E$.

Q.E.D

Corollary 4.22. *Let* $\pi : \tilde{S} \to S$ *be the blowing-up of* S *at a smooth point* p, $K_{\tilde{S}}$ *and* K_S *be the canonical divisors. Then*

$$K_{\tilde{S}}^2 = K_S^2 - 1.$$

4.3 Positivity of line bundles

Definition 4.23. A line bundle L on a complex projective surface S is called very ample if there is an embedding $\varphi : S \to \mathbb{CP}^N$ such that $L = \varphi^*\mathcal{O}_{\mathbb{CP}^N}(1)$. A line bundle D is called ample if $D^{\otimes k}$ is very ample.

Theorem 4.24. *Let* L *be a line bundle on a complex surface* S. *Then* L *is very ample if the following holds*

1. *(Base point free)* $H^0(L) \to H^0(L \otimes \mathcal{O}_S/\mathfrak{m}_p)$ *is surjective for any* p.
2. *(Separate points)* $H^0(L) \to H^0(L \otimes \mathcal{O}_S/\mathfrak{m}_p) \oplus H^0(L \otimes \mathcal{O}_S/\mathfrak{m}_q)$ *is surjective for any points* $p \neq q$.
3. *(Separate tangents)* $H^0(L) \to H^0(L \otimes \mathcal{O}_S/\mathfrak{m}_p^2)$ *is surjective for any point* p.

Lemma 4.25. *Let* L *be an ample line bundle, then* $c_1(L) > 0$.

Proof. By taking a power, we may assume that L is very ample. Hence

$$c_1(L) = \varphi^*(c_1(\mathcal{O}_{\mathbb{CP}^N}(1))) > 0.$$

Q.E.D

Ampleness has many characterizations. The following numerical criterion is very useful.

Theorem 4.26 (Nakai-Moishezon criterion)**.** *Let L be a line bundle on a complex projective surface S. Then L is ample if and only if $L \cdot C > 0$ for any irreducible curve C and $L^2 > 0$.*

To prove the theorem, an ideal is to show that $|kL|$ defines a finite morphism $S \to \mathbb{P}^N$. For that, we need the following theorem and (asymptotic) Riemann-Roch Theorem.

Theorem 4.27 (Serre's ampleness criterion)**.** *Let L be a line bundle on a complex projective variety X. Then L is ample if and only if for any coherent sheaf $\mathcal{F}$ there is an integer $m_0 = m(\mathcal{F})$ such that we have $H^p(L^k \otimes \mathcal{F}) = 0$ for $p > 0$ and all $k \geq m_0$.*

Proof. It is clear that vanishing of $H^1(L^k \otimes \mathcal{F})$ for any coherent sheaf $\mathcal{F}$ and $k \gg 0$ implies that L is ample. Assume that L is ample, we are going to show that for any coherent sheaf $\mathcal{F}$ there exists an integer $m_0 = m(\mathcal{F})$ such that $H^p(L^k \otimes \mathcal{F}) = 0$ for $p > 0$ and all $k \geq m_0$. We may assume that $X = \mathbb{CP}^N$. Let $0 \to \mathcal{F}_1 \to \mathcal{E} \to \mathcal{F} \to 0$ be an exact sequence, where $\mathcal{E}$ is free. Then $H^i(E(k)) = 0$ for $k \gg 0$ which implies that $H^i(\mathcal{F}(k)) = H^{i+1}(\mathcal{F}_1)$. Note that $H^{N+1}(\mathcal{F}_1) = 0$ by Cěch cohomology. By induction on i, we can draw our conclusion. Q.E.D

5 Branched coverings

In this section, we will recall definition and basic properties of varieties and branched covering. A comprehensive reference is [GR84].

5.1 Complex varieties

Algebraic varieties as well as analytic varieties are generalizations of complex manifolds. In the notes, we consider complex projective varieties and always work over the complex field $\mathbb{C}$.

Definition 5.1 (Algebraic varieties)**.** A closed algebraic subset X of $\mathbb{C}^n$ is a set of common solutions of finitely many polynomials, i.e. $X = \{x \in \mathbb{C}^n \mid f_1(x) = \cdots = f_m(x) = 0\}$, where $f_1, \ldots, f_m$ are polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. A closed algebraic subset X of $\mathbb{C}^n$ is called an affine algebraic variety if it is irreducible, i.e. if $X = X_1 \cup X_2$, where X_1 and X_2 are closed algebraic subset, then $X = X_1$ or $X = X_2$. An irreducible topological space X is an algebraic variety if it has a finite cover $\bigcup_{i=1}^m U_i$ such that U_i are affine varieties and the diagonal morphism $\Delta : X \to X \times X$ is closed.

An algebraic variety by definition is equipped with a topology called Zariski topology whose open sets are complements of closed algebraic subsets. Zariski topology in general is not Hausdorff but has its own advantage. Comparing with usual topology, Zariski topology is very coarse. For instance, the closed subsets of the affine line $\mathbb{C}$ under Zariski topology are just sets of finitely many points.

In analytic setting, analogues to algebraic varieties can be developed.

Definition 5.2 (Analytic Varieties)**.** Let $U \subset \mathbb{C}^n$ be an open subset. An analytic subset of U is a closed subset $X \subset U$ such that for any $x \in X$ there exists an open neighborhood $x \in V \subset U$ and finitely many holomorphic functions $f_1, \ldots, f_m$ such that $X \cap V = \{z \mid f_1(z) = \cdots = f_m(z) = 0\}$. An irreducible analytic subset is called an affine analytic variety. An analytic variety is irreducible second-countable Hausdorff topological space X which has a open covering $\bigcup_{i=1}^k U_i$ such that each U_i is biholomorphic to an affine analytic variety V_i.

Example 5.3. Let $X \in \mathbb{C}^n$ is a complex manifold. Then X is an analytic variety.

Example 5.4 (Projective spaces). Consider the space $\mathbb{P}^n$ of lines in $\mathbb{C}^{n+1}$ passing through the origin, i.e. $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*$, where $\mathbb{C}^*$ acts on

$\mathbb{C}^{n+1}$ by multiplication. It is covered be n-dimensional affine spaces $U_i = \{[x_0, \ldots, x_n] \in \mathbb{P}^n \mid x_i \neq 0\}$. One can check that $\Delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ is closed (or equivalently in analytic setting, $\mathbb{P}^n$ is Hausdorff). The space $\mathbb{P}^n$ is called the projective space of dimension n.

Definition 5.5. A subvariety $Y \subset X$ is an irreducible closed algebraic subset of X. An algebraic variety is called a projective variety if it is isomorphic to a subvariety of the projective space $\mathbb{P}^n$.

Example 5.6. Let C be the closed subset defined by $xw - yz = 0$, $z^2 - yw = 0$, $y^2 - xz = 0$ in $\mathbb{P}^3$. Then C is a projective variety of dimension one.

Analytic theory and algebraic theory of projective varieties are closely related.

Theorem 5.7 (Chow's Theorem [Cho49])**.** *A complex submanifold of $\mathbb{P}^n$ is an projective algebraic variety.*

Proof. See for example [GH94], page 167. Q.E.D

Theorem 5.8 (GAGA [Ser56])**.** *Let X and Y be projective varieties over $\mathbb{C}$, and X^{an} and Y^{an} be the analytifications. Then*

1. *Every holomorphic map $X^{an} \to Y^{an}$ is algebraic.*
2. *Every coherent analytic sheaf $\mathcal{F}^{an}$ over X^{an} is algebraic.*
3. *$H^i(X, \mathcal{F}) = H^i(X^{an}, \mathcal{F}^{an})$.*

Chow's theorem and GAGA provide an important philosophical principle of treating (projective) algebraic and analytic varieties as the same.

Definition 5.9. A point x in a variety X is called a smooth point if the local ring $\mathcal{O}_{X,x}$ is regular, i.e. $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = \dim X$, where $\mathfrak{m}$ is the unique maximal ideal of the local ring $\mathcal{O}_{X,x}$. A variety X is smooth if every point of X is smooth.

Smoothness is a local condition. Given a point $o \in X$, we may assume that X is an subvariety of $\mathbb{C}^n$ defined by finitely holomorphic functions $f_1, \ldots, f_t$. Then X is smooth at o if and only if $\operatorname{rank}(Jac(f_1, \ldots, f_t)(o)) = n - \dim X$.

Example 5.10. Let C be a curve defined by $x^2 = y^2$ in $\mathbb{C}^2$. Then C is singular at the origin $(0,0)$.

Example 5.11. The curve C defined by $xw - yz = 0$, $z^2 - yw = 0$, $y^2 - xz = 0$ in $\mathbb{P}^3$ is smooth. It is isomorphic to $\mathbb{P}^1$. For example, locally at $o = [1,0,0,0] \in \mathbb{P}^3$, the curve C is isomorphic to the curve defined by $f_1 = z - x^2 = 0$, $f_2 = y - z^3 = 0$ in $\mathbb{C}^3$ with o being identified with $(0,0,0)$. One can check that the Jacobian matrix at 0 has rank $2 = 3 - \dim C$.

Definition 5.12. An algebraic variety X is normal, if for any point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is an integrally closed domain.

Example 5.13. Smooth varieties are normal.

Theorem 5.14 (Serre's criterion for normality)**.** *An algebraic variety X is normal if and only if X is* R_1*, i.e.* codim $\mathrm{Sing}(X) \geq 2$*, and* S_2*, i.e. every regular function on $X - \mathrm{Sing}(X)$ extends to a regular function on X.*

Proof. See for example [Mat89] Theorem 23.8 on page 183, or The Stack Project Lemma 10.144.4. Q.E.D

Corollary 5.15. *An algebraic curve C is normal if and only if it is smooth.*

Another useful characterization of normality is that the varieties is locally unibranched, i.e. locally it has only one irreducible component.

Theorem 5.16 (Normal⇒Unibranch)**.** *Let $x \in X$ be a normal point. Then X is irreducible at x.*

Proof. See [GR84] page 125. Q.E.D

Example 5.17. Let C be the curve defined by $xy = 0$ in $\mathbb{C}^2$. Then C is not normal.

Example 5.18 (Isolated non-normal surface singularity). Using Theorem 5.16, one can construct an isolated non-normal surface singularity in the way by gluing two points on a smooth surface. For example, let S be the surface obtained from the complex plane $\mathbb{C}^2$ be gluing two points p and q. Then S is not normal. More precisely, S is the surface defined by the algebra $A = \{f(x,y) \mid f(p) = f(q)\}$. For example, let $p = (0,1)$ and $q = (0,-1)$, then $A = k[x, xy, y^2, y - y^3]$.

5.2 Structure of branched coverings

In this subsection, we will discuss branched coverings. Two good references are [Roa79] and [Nam87]. We start by recalling the definition of a finite holomorphic map.

Definition 5.19. Let $f : X \to Y$ be a holomorphic map between two analytic varieties. We say f is a finite map if it is closed with finite fibers.

In many books on analytic varieties, branched coverings are treated as a special case of analytic coverings. In our case, we don't need too much generality on analytic coverings.

Definition 5.20 (Branched covering)**.** Let Y be a smooth complex manifold. A finite branched covering of Y is a normal analytic variety X together with a proper surjective holomorphic map $f : X \to Y$ with finite fibers.

Example 5.21. Let S be the Fermat surface $x_0^d + x_1^d + x_2^d + x_3^d = 0$ in $\mathbb{P}^3$. Consider the restriction map $\pi : S \to \mathbb{P}^2$ of the central projection $\pi' : \mathbb{P}^3 \setminus \{[1,0,0,0]\} \to \mathbb{P}^2$, i.e. $\pi'([x_0, x_1, x_2, x_3]) = [x_1, x_2, x_3]$. Then π is a branched covering.

Note that the above definition is the same as saying that f is a finite holomorphic map (or finite morphism in algebraic geometry). In fact, we have the following characterization of proper maps.

Theorem 5.22. *Let $f : X \to Y$ be a holomorphic map between analytic varieties. Then f is proper if and only if f is closed and fibers are compact.*

Proof. See [GR84] Proposition in page 175. Q.E.D

Corollary 5.23. *A proper holomorphic map with discrete fibers is a finite map.*

The following Proposition reveals the local structure of a holomorphic finite map.

Proposition 5.24. *Let $f : X \to Y$ be a holomorphic finite map between two analytic varieties. For any $y \in Y$ with the fiber $f^{-1}(y) = \{x_1, \dots, x_r\}$, there exists disjoint open neighborhoods U_i of x_i and an open neighborhood V of y such that*

1. *$W_i = U_i \cap f^{-1}(V)$ are disjoint;*
2. *$\bigcup_{i=1}^r W_i = f^{-1}(V)$;*
3. *$W_i \to V$ is a finite cover.*

Proof. (1) and (2) follow from that a finite map is open (using open mapping theorem, see Page 107 [GR84]). To show (3), only need to show that $W_i \to V$ is closed. Let A be an analytic subset in W_i. We can take an analytic subset $B \subset X$ such that $B \cap W_i = A$. Then $f(A) = f(B \cap W_i) = f(B) \cap f(W_i)$ is a closed subset of V. Q.E.D

Definition 5.25. Denote by $R_f = \{x \in X \mid f \text{ is not locally biholomorphic at } x\}$ and $B_f = \pi(R_f)$. We call R_f and B_f ramification locus and branch locus respectively.

Note that $\dim X = \dim Y$ since π is finite. Given a point $p \in X$, if f is not biholomorphic, then the Jacobian $J(f)$ of f vanishes at p. Recall the Jacobian locally is the holomorphic function $\det\left(\frac{\partial y_i(\pi)}{\partial x_j}\right)$, where $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are local coordinates of small open neighborhoods $f^{-1}(V)$ of p and V of $f(p)$. We see that R_π is an closed analytic subset of X and so is B_f.

When X is also smooth, one see that $R_f = \operatorname{Supp} \Omega^1_{X/Y}$, where Ω^1_X and Ω_Y are the cotangent bundles, and $\Omega^1_{X/Y} = \Omega^1_X / f^*\Omega_Y$.

One of the famous result on R_f is the purity theorem.

Theorem 5.26 (Purity of branch locus)**.** *The branch locus B_f is a reduced divisor defined by the Jacobian $J(f)$.*

The analytic version of the purity theorem was proved by Grauert and Remmert [GR55]. The algebraic analogues and generalizations was proved by Zariski, Nagata and some other algebraic geometers (see [Zar58, Nag58] etc.). The papers by Altman and Kleiman [AK71, AK73] provides an elementary proof.

Example 5.27. Let S be the Fermat cubic surface $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$. Consider the map $\pi : S \to \mathbb{P}^2$ given by $\pi([x_0, x_1, x_2, x_3]) = [x_1, x_2, x_3]$. The branch locus B_π is nothing but the Fermat cubic curve $x_1^3 + x_2^3 + x_3^3 = 0$ in $\mathbb{P}^2$.

By Purity theorem, we know that $\pi : X \setminus R_\pi \to Y \setminus B_\pi$ is simply a finite unramified covering (i.e. a topological covering). Conversely, such a unramified covering extends uniquely to a branched covering by Riemann Extension Theorem (see for instance [GR84], [GH94] or [GR09]).

Theorem 5.28 (Riemman Extension Theorem)**.** *Let X be a normal analytic variety, T be an analytic subset of X of codimension* $\operatorname{codim} T \geq 1$ *and f be a holomorphic function on $X \setminus T$. Assume that* $\operatorname{codim} T \geq 2$ *or f is locally bounded in T, i.e. there is an open polydisc $\Delta(t, r) \subset X$ for any point $t \in T$ such that f is bounded on $\Delta(t, r) \setminus T$. Then f extends uniquely to a holomorphic function on X.*

In Riemman extension theorem, locally boundedness is important. Let C be curve defined by $x^2 - y^3 = 0$ in $\mathbb{C}$. Then $f = xy^{-1}$ is holomorphic on $C \setminus \{0\}$ and has a unique extension $f(0) = 0$. But it is not holomorphic on C. In fact, if f is continuos at 0, then f would be holomorphic.

Definition 5.29. Two branched coverings $\pi_1 : X_1 \to M$ and $\pi_2 : X_2 \to M$ are isomorphic if there is a biholomorphic map $\varphi : X_1 \to X_2$ such that the following diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi} & X_2 \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & M & \end{array}$$

commutes.

Proposition 5.30 (Extensions of unramified coverings)**.** *Two branched coverings $\pi : X \to Y$ and $\pi' : X' \to Y$ with $B_\pi = B_{\pi'}$ are isomorphic, if and only if $\pi : X \setminus \pi^{-1}(B_\pi) \to Y \setminus B_\pi$ and $\pi' : X' \setminus \pi'^{-1}(B_{\pi'}) \to Y \setminus B_{\pi'}$ are isomorphic.*

Proof. The necessity is clear. Assume that there is a biholomorphic map $\rho : X \setminus \pi^{-1}(B_\pi) \to X' \setminus \pi^{-1}(B_{\pi'})$ such that $\pi = \pi' \circ \rho$. We want to show that ρ extends to a biholomorphic map $\tilde{\rho} : X \to X'$. For any smooth point $x \in X$, choose a connected small open neighborhood V of $\pi(x)$. Let U be the unique connected component of $\pi^{-1}(V)$ such that $U \cap \pi^{-1}(\pi(x)) = x$. Let z be a point in U such that $y = \pi(z) \in V \setminus B_\pi$. Then there is an unique connected component W of $\pi'^{-1}(V)$ such that $\rho(z) \in W$ and $W \cap \pi'^{-1}(\pi(x)) = \{x'\}$. We can define $\tilde{\rho}(x) = x'$. It is clear that $\tilde{\rho}$ is continuous. By Riemman Extension Theorem, ρ can be extended to a holomorphic map $X - \mathrm{Sing}(X) \to X'$. Since X is normal, codim $\mathrm{Sing}X \geq 2$. Then $\tilde{\rho}$ can be further extended uniquely to a holomorphic map $\rho : X \to X'$. Conversely, ρ^{-1} extends to a unique holomorphic map $\rho^{-1} : X' \to X$. Therefore, $\tilde{\rho}$ is an isomorphism between the two branched coverings. Q.E.D

There are some immediate consequence of Proposition 5.30.

Definition 5.31. Denote by G_π the group of automorphisms of $\pi : X \to M$. We call $\pi : X \to M$ a Galois covering if G_π acts transitively on every fiber of π, i.e. for any $x_1, x_2 \in \pi^{-1}(y)$ there exists a $\sigma \in G_\pi$ such that $\sigma(x_1) = x_2$. A Galois covering $\pi : X \to M$ is called an Abelian covering if G_π is an Abelian group.

Corollary 5.32. *A branched covering $\pi : X \to Y$ is Galois if and only if $\pi' : X - \pi^{-1}(B_\pi) \to Y - B_\pi$ is Galois. Moreover, $G_\pi = G_{\pi'}$.*

Proof. Necessity is clear. Assume that $\pi' : X - \pi^{-1}(B_\pi) \to Y - B_\pi$ is Galois. We want to show that $\pi : X \to Y$ is Galois. It suffices to show that for any point $y \in B_\pi$ and points $x_1, x_2 \in \pi^{-1}(y)$ there is a transformation $\sigma \in G_\pi$ such that $\sigma(x_1) = x_2$. Let U_i be disjoint connected open neighborhoods of x_i such that $U_i \cap \pi^{-1}(y) = x_i$, where $i = 1, 2$. Then there is $\sigma_1 \in G_{\pi'}$ such that $\sigma_1(U_1 \setminus \pi^{-1}(B_\pi)) = U_2 \setminus \pi^{-1}(B_\pi)$. By Proposition 5.30, then σ' extends uniquely to a covering transformation σ and $\sigma(x_1) = x_2$. Q.E.D

The following theorem gives explicit description of local structure of a branched covering.

Theorem 5.33 (Ramification index)**.** *Let $\pi : X \to Y$ be a branched covering, y be a smooth point of B_π and x be a point in $\pi^{-1}(y)$ which is indeed smooth in X. Then there exist small open neighborhoods V of y and U of x with local coordinate systems $(y_1, \ldots, y_n)$ and $(x_1, \ldots, x_n)$ respectively such that*

1. *$U \cap \pi^{-1}(y) = \{x\}$,*
2. *$B_\pi \cap V = \{y_1 = 0\}$, and*
3. *$\pi : U \to V$ is defined by $\pi(x_1, \ldots, x_n) = (x_1^e, \ldots, x_n)$, where e is an integer.*

Proof. By Proposition 5.24 and purity theorem, we know the existence of U and V such that (1) and (2) holds. Let $(z_1, \ldots, z_n)$ be the coordinates of $U \subset \mathbb{C}^n$. Since π is locally branched along a smooth divisor defined by $y_1 = 0$.We may assume that the ramification locus is defined by $z_1 = 0$. Then we can write

$$\pi(t, z) = (ut^e, f_1(z) + tg_1(t, z), \ldots, f_{n-1}(z) + tg_{n-1}(t, z)),$$

where $t = z_1$, $z = (z_2, \ldots, z_n)$, u is a unit, and f_i and g_i are non-constant analytic functions. Since π ramifies only along $t = 0$, then the Jabobian of π is $J(\pi) = vt^m$, where v is a unit. Let w be a unit such that $w^e = u$. We consider the covering map $\pi^1 : \mathbb{C}^n \to \mathbb{C}^n$ given by $\pi^1(t, z) = (wt, f_1(z) + tg_1(t, z), \ldots, f_{n-1}(z) + tg_{n-1}(t, z))$. It is then unramified. Therefore, $J(\pi^1) = v'$, where v' is a unit. Therefore, we can take $(wt, f_1(y) + tg_1(t, y), \ldots, f_{n-1}(y) + tg_{n-1}(t, y))$ as a coordinate system of U.

Q.E.D

Remark 5.34. Let $\pi : X \to Y$ be a branched covering and $x \in X$ be a point such that the branch locus B_π in a local neighborhood U of $y = \pi(x)$ is a simple normal crossing divisor, i.e. $B_\pi \cap U$ can be defined by $y_1 \cdots y_d = 0$, where $(y_1, \ldots, y_n)$ is a coordinate system of U and $d \leq n$. Then repeating the above proof, one see that there is an open neighborhood V of x such that $V \cap \pi^{-1}(y) = x$ and $\pi : V \to U$ is given by $\pi(x_1, \ldots, x_n) = (x_1^{e_1}, \ldots, x_d^{e_d}, x_{d+1}, \ldots, x_n)$.

Corollary 5.35. *Let C be an irreducible component of B_π and $C' = C -$ $\mathrm{Sing}(C)$. Then $e_x = e_y$ for any $x, y \in C'$, where e_x is the integer defined in Theorem 5.33.*

Definition 5.36. Let C be an irreducible component of B_π and $y \in C$ is a smooth point. We call e_y the ramification index of π along C.

Definition 5.37. Assume that $B_\pi = B_1 + \cdots + B_r$ is the branched locus of $\pi : Y \to M$. The ramification index along B_i is defined as $e_i = e_y$, where y is a general point in B_i.

Definition 5.38. Let $\pi : X \to M$ be a branch covering with ramification locus $R_\pi = R_1 + \cdots + R_r$, where R_i are irreducible components of R. The ramification divisor of π is the divisor $R = \sum(e_i - 1)R_i$, where e_i is the ramification index of π along $\pi(R_i)$.

Theorem 5.39 (Relative canonical divisor)**.** *Let $\pi : X \to M$ be a branch covering. Then*

$$K_X = \pi^* K_M + R.$$

Proof. It is clear that Supp $(K_X - \pi^* K_M) = R_\pi$. To determine the coefficient of each irreducible component of R, we work locally. Let R_i be an irreducible component and $B_i = \pi(R_i)$. Note that the branched covering is locally given by $\pi(x_1, \dots, x_n) = (x_1^e, \dots, x_n)$. Thus, around R_i, the $J(\pi) = e_i x_1^{e_i - 1}$. Therefore, $K_X - \pi^* K_M = R = \sum_i (e_i - 1)R_i$. Q.E.D

5.3 Algebraic function fields and Galois Coverings

In this subsection, we will assume that $\pi : X \to Y$ is a branched covering between complex projective varieties so that the meromorphic function field $\mathbb{C}(X)$ is a finite extension of the meromorphic function field $\mathbb{C}(Y)$.

Theorem 5.40 (Topology Galois group=Galois group of extension)**.** *Let $\pi : X \to Y$ be a branched covering between complex projective varieties. Then it is Galois if and only if $\mathbb{C}(X)$ is a Galois extension of $\mathbb{C}(Y)$. Moreover, if π is Galois, then*

$$G_\pi = \mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)).$$

Proof. By Corollary 5.32, we know that $\pi : X \to Y$ is Galois if and only if $\pi' : X \setminus \pi^{-1}(B_\pi) \to Y \setminus B_\pi$ is Galois and $G_\pi = G_{\pi'}$. Since B_π and $\pi^{-1}(B_\pi)$ are divisors, then $\mathbb{C}(X) = \mathbb{C}(X \setminus \pi^{-1}(B_\pi))$ and $\mathbb{C}(Y) = \mathbb{C}(Y \setminus B_\pi)$. It suffices to show that $\pi' : X \setminus \pi^{-1}(B_\pi) \to Y \setminus B_\pi$ is Galois if and only if $\mathbb{C}(X \setminus \pi^{-1}(B_\pi))$ is a Galois extension of $\mathbb{C}(Y \setminus B_\pi)$.

Let π be a Galois covering. It is clear that for $\sigma \in G_\pi$, $\sigma^*\pi^*h = \pi^*h$ for any $h \in \mathbb{C}(Y \setminus B_\pi)$. Since G_π acts transitively on X, for any f in $\mathbb{C}(X \setminus \pi^{-1}(B_\pi))$ such that $f \circ \sigma = f$, we obtain a meromorphic function $g \in \mathbb{C}(Y \setminus B_\pi)$ which is given by $g(x) = f(y)$, where $y \in \pi^{-1}(x)$. It then follows that

$$\mathbb{C}(Y \setminus B_\pi) = \{f \in \mathbb{C}(X \setminus \pi^{-1}(B_\pi)) \mid f \circ \sigma = f \text{ for any } \sigma \in G_\pi\}.$$

Since $G_\pi = G_{\pi'}$ is finite, then by a theorem of Artin, we know that $\mathbb{C}(X \setminus \pi^{-1}(B_\pi))/\mathbb{C}(Y \setminus B_\pi)$ is a Galois extension with Galois group $G_\pi = G_{\pi'} = \mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y))$.

Conversely, assume that $\mathbb{C}(X)/\mathbb{C}(Y)$ is a Galois extension. Then there exist a minimal irreducible (because X is connected) polynomial $p(T) \in \mathbb{C}(Y)[T]$ such that $\mathbb{C}(X) = \mathbb{C}(Y)(R_1, \ldots, R_d)$, where R_i are the distinct roots of $p(T)$. By clearing denominators, we get a polynomial $p'(T)$ is defined over $\mathbb{C}[Y]$ and we may assume that $X = \{p'(T) = 0\} \subset Y \times \mathbb{C}$. Then outside the divisor Δ defined by the discriminant (whose support is exactly the branch locus), the finite morphism $X \setminus \pi^{-1}(\Delta) \to V$ is an unranmified covering. Then each element $\tau \in \mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y))$ induces an automorphism of U which fixed V. Therefore, $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y))$ can be viewed as subgroup of G_π. This implies that for any R_i and R_j, there is a $\sigma \in G_\pi$ such that $\sigma(R_i) = R_j$. We conclude that π is Galois. Q.E.D

Another proof can be found in [Roa79]. The essence of the proof is that a holomorphic map between two branched covering of complex projective varieties is algebraic. Therefore, a equivalence between two branched covering corresponds to an element in the Galois group of the extension of function fields.

This result tells us a way to construct a branched covering associated to a finite field extension.

Theorem 5.41 (Construction of branched coverings). *Let Y be a smooth projective manifold and K be a finite extension of the function field $\mathbb{C}(Y)$. Then there exists an branched covering $\pi : X \to Y$ such that $\mathbb{C}(X) = K$.*

Proof. Let $p(z) = z^d + \eta_1 z^{d-1} + \cdots + \eta_d \in \mathbb{C}(Y)[z]$ be the minimal polynomial of α. Let L be the minimal divisor on X such that $\mathrm{div}(p_i) + iL \geq 0$. Denote by $\mathcal{L} = \mathcal{O}_X(L)$. Let l be a rational section of $\mathcal{L}$. Then $f_i = \eta_i l^i$ is global section of $\mathcal{L}$ by the choice of L. Consider the variety X_0 defined by $f(z) = z^n + p^* f_1 z^{n-1} + \dots p^* f_d = 0$ in the total space $\mathbb{L}$ of $\mathcal{L}$ with projection $p : \mathbb{L} \to X$. By minimality, $f(z)$ is a minimal polynomial for $p^* l \cdot \alpha$. By the construction of X_0, we see that $\mathbb{C}(X_0) = K$. By taking normalization of X_0, we get a branched covering $\pi : X \to Y$ with X a normal variety. Q.E.D

Remark 5.42. Note that the normal variety X may not be defined by a single polynomial on $\mathbb{L}$.

Proposition 5.43. *Let $\pi : X \to Y$ be a Galois covering with Galois group G_π. Then X/G_π is canonically isomorphic to Y.*

Proof. First, notice that $c : X/G_\pi \to Y$ is bijective and continuous map. Take a homomorphic function f on Y, then $\pi^* f \circ \sigma = \pi^* f$ for any $\sigma \in G_\pi$. We define a function $g(y) = \pi^* f(x)$ on X/G_π, where $x \in \pi^{-1}(y)$. It is then holomorphic, since $\pi^* f$ is holomorphic. Therefore, the map c is a holomorphic map. Hence, it is a biholomorphic map. Q.E.D

Let S be a smooth algebraic surfaces with a finite group G acting transitively on S, then S/G has only ADE singularities.

Theorem 5.44 (Criterion for smoothness). *Let $\pi : X \to Y$ be a branched covering with branch locus $B_\pi = B_1 + \cdots + B_r$. Assume that B_i are smooth and B is a simple normal crossing divisor. Then X is smooth.*

Proof. It is clear if $x \in X \setminus \pi^{-1}(B_\pi)$, then x is smooth. Now let $x \in \pi^{-1}(B_\pi)$. If $\pi(x)$ is a smooth point in B_π, then by Theorem 5.33, locally, X is defined by $t = z_1^n$ in $U \times \mathbb{C}$, where z_1 defines B_π in an open neighborhood U of $\pi(x)$. Hence X is smooth at x. Similar, if $x \in B_{i_1} \cap \cdots \cap B_{i_r}$, then locally, X is defined by $t_i = z_i^{n_i}$, $i = 1, \dots, r$ in $U \times \mathbb{C}^r$. Hence, X is smooth. Q.E.D

Theorem 5.44 together with the existence of smooth resolution of singularities implies the following useful theorem.

Theorem 5.45 (Canonical resolution)**.** *Let $\pi : X \to Y$ be a branched covering with branch locus $B_\pi = B_1 + \cdots + B_r$. Let $\sigma : \tilde{Y} \to Y$ be an embedded resolution of B. Then the fiber product $Y = X \times_Y \tilde{Y}$ is smooth and birational to X.*

Example 5.46 (Cyclic cover)*.* Let M be a smooth projective manifold and L be a line bundle on M. Given a section $s \in \Gamma(M, L)$, we have an algebraic variety X in the total space $\mathbb{L} = \text{Spec } (\oplus L^{-i})$ of L defined by $z^n = s$, where z is the coordinate of the fiber $\mathbb{C}$. Then $\tilde{X} \to M$ is a cyclic cover of degree n and $\mathbb{C}(X) = \mathbb{C}(M)[s^{1/n}]$, where $\tilde{X}$ is the normalization of X.

Example 5.47 (Abelian covers)*.* A branched covering $\pi : X \to Y$ is an Abelian covering if and only if $\mathbb{C}(X) = \mathbb{C}(Y)[\xi_1, \ldots, \xi_k]$ such that $\xi_1^{n_1} = \eta_1, \ldots, \xi_k^{n_k} = \eta_k$, where $\eta_i \in \mathbb{C}(Y)$.

6 Constructions of ball quotient surfaces

In this chapter, we present ball quotient surfaces by following Hirzebruch (see also [Nam87] pages 75-80.).

A line arrangement $\mathcal{A} = \{L_1, \ldots, L_k\}$ is a finite collection of lines in $\mathbb{P}^2$. Denote by $l_i = 0$ the defining equations of the lines. The polynomial $Q(\mathcal{A}) = l_1 \cdots l_k$ is called the defining polynomial of the line arrangement $\mathcal{A}$. A point $p \in L = \bigcup_{L_i \in \mathcal{A}} L_i \subset \mathbb{P}^2$ is a called r-fold point of $\mathcal{A}$ if the order $r_p = \text{ord}_p Q(\mathcal{A}) = r$. An r-fold point is called a multiple point if $r \geq 3$. Denote by t_r the number of r-fold points of $\mathcal{A}$.

Assume that $\mathcal{A}$ is not a pencil, i.e. $L_1 \cap \cdots \cap L_k = \emptyset$. We consider the algebraic set X defined by $z_i^n l_j = z_j^n l_i$, $1 \leq i, j \leq k$, in $\mathbb{P}^2 \times \mathbb{P}^{k-1}$, where $[z_1, \ldots, z_k]$ is the homogeneous coordinate of $\mathbb{P}^{k-1}$. Consider the morphism $\varphi : \mathbb{P}^2 \to \mathbb{P}^{k-1}$ define by $\varphi([x, y, z]) = [l_1, \ldots, l_k]$ and the morphism $\alpha : \mathbb{P}^{k-1} \to \mathbb{P}^{k-1}$ defined by $\alpha([z_1, \ldots, z_k]) = [z_1^n, \ldots, z_k^n]$. Then X is the fiber product $\mathbb{P}^2 \times_{\mathbb{P}^{k-1}} \mathbb{P}^{k-1}$. We see that X is a degree

n^{k-1} Abelian covering of the projective plane branched along the union of the lines $L_1 \cup \cdots \cup L_k$.

The singularities of X can be resolved by blowing up $\mathbb{P}^2$ at those multiple points of $\mathcal{A}$ and take fiber product.

$$\begin{array}{ccccc} Y & \xrightarrow{\rho} & X = \mathbb{P}^2 \times_{\mathbb{P}^{k-1}} \mathbb{P}^{k-1} & \longrightarrow & \mathbb{P}^{k-1} \\ \downarrow \sigma & & \downarrow \pi & & \downarrow \alpha \\ \widehat{\mathbb{P}^2} & \xrightarrow{\tau} & \mathbb{P}^2 & \xrightarrow{\varphi} & \mathbb{P}^{k-1} \end{array}$$

Using additivity of cohomology, we know that the Euler number of Y is $e(Y) = e(X - \mathrm{Sing}(X)) + e(\rho^{-1}(\mathrm{Sing}(X)))$. The Euler number $e(X - \mathrm{Sing}(X))$ is easier to calculate. In fact,

$$\begin{aligned} e(X - \mathrm{Sing}(X)) &= \deg(\pi)e(\mathbb{P}^2 - L) + \sum_{i=1}^{k} \deg(\pi \mid_{\pi^{-1}(L_i)})e(L_i - \mathrm{Sing}(L)) \\ &\quad + \sum_{r_p=2} \deg(\pi \mid_{\pi^{-1}(p)}) \\ &= n^{k-1}e(\mathbb{P}^2 - L) + n^{k-2}e(L - \mathrm{Sing}(L)) + n^{k-3}t_2 \\ &= n^{k-1}(3 - 2k + \sum_{r\geq 2}(r-1)t_r) \\ &\quad + n^{k-2}(2k - \sum_{r \geq 2} rt_r) + n^{k-3}t_2. \end{aligned}$$

To calculate $e(\rho^{-1}(\mathrm{Sing}(X)))$, we have to know what is $\rho^{-1}(\mathrm{Sing}(X))$. Let p be a r-fold point and $E_p = \tau^{-1}(p) \subset \widehat{\mathbb{P}^2}$. Then $\sigma^{-1}(E_p) = \rho^{-1} \circ \pi^{-1}(p)$. Note that $\pi^{-1}(p)$ consists of n^{k-r-1} distinct points. So $\sigma^{-1}(E_p)$ consists of n^{k-r-1} disjoint smooth curves which are isomorphic to each other. Moreover, each curve C is a degree n^{r-1} covering of E_p with r ramification points each with ramification index $n^{r-1-1} = n^{r-2}$. In fact, assume that $L_1 \cap \cdots \cap L_r = [0,0,1] = o$, and $l_1 = x$, $l_2 = y$ and $l_i = x + a_i y$ for $3 \leq i \leq r$ and $a \neq 0$. Let t be the affine coordinate $E_p \subset Bl_o\mathbb{P}^2$ such that $Bl_o\mathbb{P}^2$ is defined by $y = tx$ in $\mathbb{C} \times \mathbb{C}^2$ with coordinates $(t,(x,y))$. Take (t,x) as the coordinates of $Bl_o\mathbb{P}^2 \subset \mathbb{C}^3$ such that $E_p = \{x = 0\}$. Then $Y \subset \mathbb{C}^2 \times \mathbb{C}^r$ is defined by $z_1^n = x$, $z_2^n = tz_1^n$ and $z_j^n = x(1 + a_j t) = (1 + a_j t)z_1^n$ for $3 \leq j \leq r$ near a point q in $\sigma^{-1}(o)$. Take (z_1, t) as the local coordinates near q, then

$\sigma(z_1, t) = (z_1^n, t)$. Therefore, the ramification index of σ along C is n. Moreover, the curve C defined by $z_1 = 0$ is a degree n^{r-1} covering of E_p, since for a general point $(t, 0) \in E_p$, there are exactly n^{r-1} points above it. Another way to see that $\sigma^*(E_p) = nC$ in a neighborhood of C is from the observation that $\rho^*(nC_1 + \cdots nC_r) + rnC = \sigma^*(\tilde{L}_1 + \cdots + \tilde{L}_r + rE_p)$, where C_i and $\tilde{L}_i$ are the preimage of L_i in a neighborhood of q and the strict transform of L_i respectively.

Using additivity of cohomology and the fact that E_p is isomorphic to $\mathbb{P}^1$, we see that $e(C) = n^{r-1}(2-r) + rn^{r-2}$. Therefore,

$$e(\rho^{-1}(\mathrm{Sing}(X))) = \sum_{r \geq 3} t_r n^{k-r-1}(n^{r-1}(2-r) + rn^{r-2})$$

Let $f_1 = \sum_{r \geq 2} r t_r$ and $f_0 = \sum_{r \geq 2} t_r$.

Proposition 6.1.

$$\frac{c_2(Y)}{n^{k-3}} = \frac{e(Y)}{n^{k-3}} = n^2(3 - 2k + f_1 - f_0) + 2n(k + f_0 - f_1) + (f_1 - t_2).$$

To calculate $c_1^2(Y) = K_Y^2$, we need to know the canonical divisor K_Y. Since Y is a branched covering of $\widehat{\mathbb{P}^2}$, Then $K_Y = \sigma^* K_{\widehat{\mathbb{P}^2}} + R$, where R is the ramification divisor of σ.

It is easy to see that $K_{\widehat{\mathbb{P}^2}} = \tau^* K_{\mathbb{P}^2} + \sum_{r_p \geq 3} E_p$.

Denote by $\widetilde{L_i}$ the birational transforms of L_i on $\widehat{\mathbb{P}^2}$. Then the ramification locus is

$$B = \sigma^*(\sum_{i=1}^{k} \widetilde{L_i} + \sum_{r_p \geq 3} E_p).$$

We have seen that

$$\sigma^* E_p = \sum_{j=1}^{n^{k-r-1}} nC_j,$$

where each C_j is a reduced irreducible curve.

Now we consider

$$\sigma^* \widetilde{L_i} = \sigma^*(\tau^* L_i - \sum_{r_p \geq 3, p \in L_i} r_p E_p).$$

Note that

$$\sigma^*\tau^*L_i = \rho^*\pi^*L_i = \rho^*\sum_{j=1}^{n^{k-2}} nL_{ij},$$

where L_{ij} are disjoint reduced irreducible curves.

Therefore, the ramification index for each irreducible component of B is n, which tells us that $R = \frac{n-1}{n}B$.

Lemma 6.2.

$$\begin{aligned} K_Y &= \sigma^*(\tau^*K_{\mathbb{P}^2} + \sum_{r_p\geq 3} E_p + \frac{n-1}{n}(\sum_{i=1}^{k}\widetilde{L_i} + \sum_{r_p\geq 3} E_p)) \\ &= \sigma^*(\tau^*K_{\mathbb{P}^2} + \sum_{r_p\geq 3} E_p + \frac{n-1}{n}(\sum_{i=1}^{k}\tau^*L_i - \sum_{r_p\geq 3} r_pE_p + \sum_{r_p\geq 3} E_p)). \end{aligned}$$

Corollary 6.3.

$$\begin{aligned} \frac{K_Y^2}{n^{k-3}} &= (n(k-3)-k)^2 - \sum_{r_p\geq 3}(n + (n-1) - (n-1)r_p)^2 \\ &= (n(k-3)-k)^2 - \sum_{r\geq 3}((2n-1)-(n-1)r)^2 t_r \\ &= (n(k-3)-k)^2 - \sum_{r\geq 2}((2n-1)-(n-1)r)^2 t_r + t_2 \\ &= n^2(-5k+9+3f_1-4f_0) + 4n(k-f_1+f_0) + f_1 - f_0 + k + t_2. \end{aligned}$$

Corollary 6.4.

$$\begin{aligned} \frac{3c_2(Y)-c_1^2(Y)}{n^{k-3}} &= n^2(f_0-k) + n(2k+2f_0-2f_1) + (2f_1+f_0-k-4t_2) \\ &= (n-1)^2(f_0-k) + (n-1)(4f_0-2f_1) + (4f_0-4t_2). \end{aligned}$$

Theorem 6.5. *For a line arrangement with $k \geq 6$ and $t_k = t_{k-1} = t_{k-2} = 0$, the surface Y is minimal and of general type for $n \geq 3$.*

Proof. It suffices to show that K_Y is nef and $K_Y^2 > 0$. Denote

$$\begin{aligned} K := &\ \tau^*K_{\mathbb{P}^2} + (2-\frac{1}{n})\sum_{r_p\geq 3} E_p + \frac{n-1}{n}(\sum_{i=1}^{k}\widetilde{L_i}) \\ &= \tau^*K_{\mathbb{P}^2} + \sum_{r_p\geq 3}(1+\frac{n-1}{n}(1-r_p))E_p + \frac{n-1}{n}\sum_{i=1}^{k}\tau^*L_i. \end{aligned}$$

By Lemma 6.2, we see that $K_Y = \sigma^* K$. Since σ is finite, it is sufficient to show that K is net and big. For any exceptional divisor E_p with $r_p \geq 3$, we see that $E_p \cdot K = -1 + \frac{n-1}{n}(r_p - 1) \geq 0$ for $n \geq 2$ and > 0 for $n \geq 3$. Let $\tilde{L}$ be a strict transformation of a line L in $\mathbb{P}^2$. Then $\tilde{L} = \tau^*(L) - \sum_{p\in L, r_p \geq 3} E_p$ and

$$\begin{aligned}\tilde{L} \cdot K &= -3 + \frac{n-1}{n}k - \sum_{p\in L, r_p\geq 3}(-1 + \frac{n-1}{n}(r_p - 1)) \\ &= -3 + (\sum_{p\in L, r_p \geq 3} \frac{2n-1}{n}) + \frac{n-1}{n}(k - \sum_{p\in L, r_p\geq 3} r_p).\end{aligned}$$

We claim that $\tilde{L} \cdot K \geq 0$ for $n \geq 3$. If there is no $p \in L$ such that $r_p \geq 3$, then $\tilde{L} \cdot K = -3 + \frac{n-1}{n}k \geq 0$. If there is only one $p \in L$ such that $r_p \geq 3$, then $\tilde{L}\cdot K = -3 + \frac{2n-1}{n} + \frac{n-1}{n}(k - r_p) \geq -3 + \frac{2n-1}{n} + \frac{3(n-1)}{n} \geq 0$. If there are at least two $p \in L$ such that $r_p \geq 3$. Then $\tilde{L}\cdot K \geq -3 + \frac{2(2n-1)}{n} + \frac{n-1}{n} > 0$. Therefore, K_Y is nef. To show that $K^2 > 0$, we may use the numerical equivalence $K_{\mathbb{P}^2} \equiv -\frac{1}{2}(L_1 + L_2 + \cdots + L_6)$. Then

$$K \equiv \sum_{r_p\geq 3}(1 + \frac{n-1}{n}(1 - r_p))E_p + (\frac{n-1}{n} - \frac{1}{2})\sum_{i=1}^{6}\tau^* L_i + \frac{n-1}{n}\sum_{i=7}^{k}\tau^* L_i.$$

Now since the coefficients are positive and $K \cdot E_p > 0$ and $K \cdot \tau^* L_i > 0$, we conclude that $K^2 > 0$.

Q.E.D

Define $F(x) = x^2(f_0 - k) + 2x(2f_0 - f_1) + 4(f_0 - t_2)$. We want to find $n \in \mathbb{N}$ such that $F(n) = 0$.

Example 6.6. Let $\mathcal{A}$ be the arrangement of 6 lines with 4 triple points and 3 double points. We have $k = 6$, $f_0 = 7$ and $f_1 = 18$. Then $F(x) = (x-4)^2$. Therefore, when $n = 5$, we obtain a surface Y which is a ball quotient surface according to Yau's theorem.

Example 6.7. Let $\mathcal{H}$ be the arrangement of 9 lines with 12 triple points. We have $k = 9$, $f_0 = 12$ and $f_1 = 36$. Then $F(x) = 3(x-4)^2$. Therefore, when $n = 5$, we get a ball quotient surface Y.

Example 6.8. Let $\mathcal{H}^*$ be the dual arrangement to $\mathcal{H}$. It has 12 lines with 9 quadruple points and 12 double points. In this case, $k = 12$, $f_0 = 21$ and $f_1 = 60$. Then $F(x) = 9(x-2)^2$.

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